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# Fundamentals of the Mechanics of Heterogeneous Media in the Circumsolar Protoplanetary Cloud: The Effects of Solid Particles on Disk Turbulence

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**Abstract**—We formulate a complete system of equations of two-phase multicomponent mechanics including the relative motion of the phases, coagulation processes, phase transitions, chemical reactions, and radiation in terms of the problem of reconstructing the evolution of the protoplanetary gas–dust cloud that surrounded the proto-Sun at an early stage of its existence. These equations are intended for schematized formulations and numerical solutions of special model problems on mutually consistent modeling of the structure, dynamics, thermal regime, and chemical composition of the circumsolar disk at various stages of its evolution, in particular, the developed turbulent motions of a coagulating gas suspension that lead to the formation of a dust subdisk, its gravitational instability, and the subsequent formation and growth of planetesimals. To phenomenologically describe the turbulent flows of disk material, we perform a Favre probability-theoretical averaging of the stochastic equations of heterogeneous mechanics and derive defining relations for the turbulent flows of interphase diffusion and heat as well as for the "relative" and Reynolds stress tensors needed to close the equations of mean motion. Particular attention is given to studying the influence of the inertial effects of dust particles on the properties of turbulence in the disk, in particular, on the additional generation of turbulent energy by large particles near the equatorial plane of the proto-Sun. We develop a semiempirical method of modeling the coefficient of turbulent viscosity in a two-phase disk medium by taking into account the inverse effects of the transfer of a dispersed phase (or heat) on the growth of turbulence to model the vertically nonuniform thermohydrodynamic structure of the subdisk and its atmosphere. We analyze the possible "regime of limiting saturation" of the subdisk atmosphere by fine dust particles that is responsible for the intensification of various coagulation mechanisms in a turbulized medium. For steady motion when solid particles settle to the midplane of the disk under gravity, we analyze the parametric method of moments for solving the Smoluchowski integro-differential coagulation equation for the particle size distribution function. This method is based on the fact that the sought-for distribution function *a priori* belongs to a certain parametric class of distributions.

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## INTRODUCTION

The material of a protoplanetary gas–dust cloud is a complex multiphase medium with regions of various densities, temperatures, and degrees of ionization. This material, generally a dust plasma, is magnetized and is in a state of strong turbulization. Understanding the evolution of a protoplanetary cloud is a necessary prerequisite for solving the question about the formation of the Earth and planets, a question intimately related to the fundamental problem of cosmogony whose solution is presently the biggest problem of science (see Schmidt, 1957; Safronov, 1982; Galimov, 2001). The planets are currently believed to be formed after the dust subdisk produced through the differential rotation of turbulized protoplanetary material in an orbit around a solar-type star and accretion when dust settles to the midplane of the disk[1] perpendicular to the rotation axis loses its gravitational stability (Toomre, 1964; Safronov, 1969, 1982; Goldreich and Ward, 1973; Nakagawa *et al.*, 1986; Makalkin, 1994; Youdin and Shu, 2002). It is now clear that the Solar planetary system was formed from the subdisk material through the formation of discrete compaction centers and their subse-

[1] The flattening of a rotating protoplanetary cloud results from the confrontation between two main dynamical forces, the gravitational and centrifugal ones. When there is an equilibrium between these forces, weaker factors, such as the thermal and viscous processes, the disk self-gravity, and the electromagnetic phenomena, become important for the evolution of the cloud. Due to the viscous forces of friction (arising from the relative shear of gas-suspension elements during their orbital motion), the disk material drifts toward the proto-Sun along a flat spiral trajectory as its angular momentum is transferred outward, from the inner disk regions to the outer ones.

quent growth (see, e.g., Safronov, 1982, 1987). It is important to emphasize that one of the key viewpoints in astrophysics regarding the origin and structure of circumstellar gas–dust accretion disks of any type is their turbulent nature (Zel'dovich, 1981; Fridman, 1989; Dubrulle, 1993; Balbus and Hawley, 1998; Richard and Zahn, 1999; Bisnovatyi-Kogan and Lovelace, 2001).

Therefore, adequate numerical simulations of the evolution of the protoplanetary cloud that surrounded the Sun at an early stage of its existence generally require taking into account the dynamical processes of the interaction between turbulized gas and dust, in particular, the modification of the carrier-phase turbulence energy by solid particles (i.e., the inverse effect of the dust component on the turbulent and thermal regimes of the disk gas component) and the influence of turbulence on the rates of phase transitions (evaporation and condensation in the cooling disk), on the jumplike disperse-particle accumulation processes (coagulation and fragmentation when particles mutually collide with one another in a flow), and, finally, on the settling of solid particles through the gas to the midplane of the disk, where they form a flattened dust layer (subdisk). In general, fine solid particles (a relatively low-inertia gas-suspension component) have a laminarizing effect on a two-phase turbulent flow (via the growth of additional dissipation), while coarse particles enhance the generation of pulsational energy via the formation of a vortex wake. It should be noted that the dust phase may be disregarded only at the initial evolutionary stage of the cosmic system under consideration when almost all of the primordial (interstellar) solid particles have already evaporated.[2] At later evolutionary stages of the protoplanetary cloud, as the disk cooled down, the dust particles condensed, their sizes increased (mainly through coagulation), and the gas dissipated from the disk system into interstellar space, the dynamical, energetic, and optical roles of the dust component increased significantly. In this case, when the disk medium is modeled, it is important to take into account the influence of dust on the flow turbulence, which is generally ambiguous and strongly depends on the volume content (concentration) and inertia of the solid particles. In particular, such effects of the dust component on the disk turbulence as the turbulent "diffusion" transfer of the disperse component attributable to the spatial nonuniformity of the dust particle distribution in the disk, the generation of additional turbulent disturbances via the collective effects related to interparticle collisions between solid particles (Shraiber *et al.*, 1980), the formation of vortex structures behind the streamlined large particles during the separation of the carrier gas flow, and the combined effect of these two flow turbulization mechanisms etc. become important at certain evolutionary stages of a heterogeneous mixture. In addition, the very presence of a polydisperse admixture in a turbulent flow complicates significantly the disk hydrodynamics, contributing to the realization of additional cosmic flow regimes. In particular, an increase in the concentration of solid particles in a heterogeneous flow related to the dust settling to the midplane of the disk under the vertical gravity of the proto-Sun leads to an additional local enhancement of the generation of turbulent flow energy attributable to the growth of the transverse relative phase velocity gradient near the midplane, i.e., to flow returbulization (cf. Goldreich and Ward, 1973).

In addition, the efficiency of the accretion mechanisms in a protoplanetary cloud (particularly at the subdisk formation stage) also depends significantly on the intensity of its turbulization; turbulence can have a completely unexpected effect on the particle coagulation in various situations, but it probably always contributes to the coagulation (Voloshchuk and Sedunov, 1975). Thus, for example, if the internal Kolmogorov turbulence scale length $\lambda_K$ is smaller than or comparable to the disperse particle size, then there is a turbulent motion of particles (similar to Brownian motion) that leads to their mutual collisions, i.e., to turbulent coagulation (which complements the effective gravitational coagulation in a quiet gas). For particles whose sizes are much smaller than $\lambda_K$, turbulence affects the evolution of fine dust through different channels. In this case, the enhancement of various coagulation processes (produced by factors other than turbulence) will result from intense turbulent particle mixing at distances larger than the Kolmogorov scale length, when the number of mutual collisions between solid particles per unit time increases significantly[3] compared to a laminar flow[4] due to chaotic turbulent pulsations. Therefore, apart from the gravitational accretion, the nongravitational accretion related, for example, to the Brownian coagulation, electric coagulation, turbulent Brownian coagulation of charged and neutral particles, etc. becomes an efficient solid particle accumulation mechanism. As the inertia of the particles increases, they will be involved in the pulsational (vortex) gas-suspension motion to a progressively lesser extent; in the long run, this leads to their effective settling to the equatorial plane of the protostar. Thus, contrary to the opinions of many research-

[2] Being constituents of the protoplanetary disk, the dust particles either evaporated when they fell into its inner high-temperature region or were preserved partially (or completely) in farther (from the Sun) and, hence, colder regions.

[3] Below, each collision between particles is assumed to lead to their coalescence (this is the so-called fast coagulation without attractive forces), but the adhesion mechanism itself is not discussed in this paper.

[4] Turbulent pulsations can contribute to drawing fine particles into the hydrodynamic wake or into the zone of action of the induction forces in the case of likely charged particles and can also promote the electrostatic coagulation via the destruction of the screening (Voloshchuk and Sedunov, 1975).

ers (see, e.g., Makalkin, 2003), turbulence of the gas–dust medium contributes in one way or another to the formation of a subdisk whose gravitational instability eventually leads to the formation of planetesimals.

Finally, there are strong arguments for the assumption that plasma effects played a significant role during the formation of the circumsolar protoplanetary disk and its early evolution. In general, the cosmic plasma is a dust plasma, i.e., contains very fine dust particles. Since any accretion disk contains solid particles of various sizes, there probably exists a boundary linear scale[5] (dependent on the electromagnetic and gravitational fields as well as on the particle charge and density) that separates the small particles comprising the dust plasma and the large particles whose motion is determined by the action of nonelectromagnetic forces. Photoelectron emission and collisions with plasma electrons and positive ions are known to be the main physical processes that determine the grain surface charge (see. e.g., Alfven and Arrenius, 1979). At the same time, a solid particle in a cosmic plasma is more frequently charged negatively to a potential of the order of several volts via collisions with electrons.[6] When an electrically conducting two-phase medium moves in an electromagnetic field, the Lorentz ponderomotive force acts on the charged particles, which gives rise to a number of additional effects, particularly during flow turbulization (see, e.g., Vereshchagin *et al.*, 1974; Busroid, 1975). In particular, such unique properties as high dissipativity, the capability for the self-organization and formation of ordered structures are characteristic of a turbulent dust plasma.

The synenergetic processes of self-organization in the thermodynamically open system of a protoplanetary cloud against the background of a large-scale shear flow of cosmic material related to its differential rotation are also a very important mechanism that shapes the properties of the cloud at various stages of its evolution, including the formation of a viscous accretion disk around the young Sun that was passing through the T Tauri stage, the formation of a dust–gas subdisk, the destruction of the latter due to gravitational instability, and the production of discrete compaction centers followed by the formation of and growth of planetesimals. This also applies to the formation of various mesoscale, relatively stable gas–dust coherent structures in the disk that probably provide the most favorable conditions for the mechanical and physical–chemical interaction between material particles (see Barge and Sommeria, 1995; Tanga *et al.*, 1996; Chavanis, 1999; Kolesnichenko, 2004). As a result, spontaneous formation and growth of a condensed dust component,[7] intensification of phase transitions and heat and mass transfer at various thermohydrodynamic parameters of the carrier and disperse phases, significant modification of the oscillation spectrum in a heavily dusted medium, etc. take place.

The protoplanetary accretion disks are known to have a significant viscosity; in combination with the differential rotation of their material, this gives rise to a permanent "intrinsic" source of thermal energy in them. Shear turbulence (Gor'kavyi and Fridman, 1994; Fridman *et al.*, 2003) and random magnetic fields (see Armitage *et al.*, 2001), with the energy of the latter being often comparable to the hydrodynamic turbulence energy,[8] are currently believed to be most likely responsible for the viscosity of differentially rotating disks.

There is extensive literature on modeling the evolution of the circumsolar protoplanetary disk without dust (see, e.g., the vast bibliography to the review paper by Bisnovatyi-Kogan and Lovelace (2001)). At the same time, the few publications on dusty disk systems cover a comparatively narrow range of problems pertaining to the problem under consideration and the results obtained in them are limited, since the turbulence models for two-phase "gas–solid particles" media discussed in them cannot be recognized to be quite satisfactory (see, e.g., Weidenschilling, 1977, 1980; Sekiya and Nakagawa, 1988; Cuzzi *et al.*, 1993; Dubrulle, 1993; Dubrulle *et al.*, 1995; Stepinski and Valageas, 1996, 1997; Goodman and Pindor, 2000; Takeuchi and Lin, 2002, 2003; Youdin and Goodman, 2004). In particular, this is because the currently existing theory of turbulence of heterogeneous flows is imperfect due to both the incompleteness of the "classical" theory of hydrodynamic turbulence and the various additional regimes of two-phase turbulent flows realized in the disk when varying the volume content and sizes of solid particles in the gas-suspension flow.

For this reason, in the presented series of papers, as applied to the problem of reconstructing the evolution of a protoplanetary gas–dust cloud, we attempt to develop a continuum model for a heterogeneous disk medium that includes the combined influence of magnetohydrodynamic and turbulence effects on the

[5] Typically, it is ~$10^{-5}$–$10^{-7}$ m.

[6] In the cases where a particle falls into a plasma region with a large number of suprathermal electrons, its negative potential can reach values of the order of several thousand volts, as a result of which it is trapped by the plasma.

[7] One of the possible scenarios for the formation and growth of dust particles in plasma consists of the following stages: first, primary clusters are formed; after the passage of a critical size, the stage of heterogeneous condensation begins; at the next stage, coagulation and agglomeration (adhesion) come to the fore; at the final stage, the surface recombination of ions, which leads to continuous cooling of the material on the surfaces of isolated multiply charged ions, becomes most important.

[8] The random magnetic fields stretching together with the accreting plasma, get mixed due to the differential rotation of the disk, and reconnecting at the boundaries between chaotic cells will also contribute significantly to the viscosity in the inner regions of the disk and in its outer atmospheric layers, where a sufficient degree of ionization of the material is reached. Large-scale magnetic fields can also play an important role in accretion physics (see Eardley and Lightman, 1975).

dynamics and heat and mass transfer in differentially rotating cosmic material by taking into account the inertial properties of a polydisperse admixture of solid particles, coagulation, and radiation. In this paper, which opens this series, we dwell mainly on the following four aspects of the problem of modeling a disk medium without touching on plasma effects:

—Formulating a basic system of mass, momentum, and energy conservation equations for the instantaneous (actual) parameters of the flow of a gas–dust mixture and radiation that are intended for numerical simulations of the circumsolar protoplanetary disk at various stages of its evolution (in particular, the laminar subdisk formation stage) and in spatial zones located at various distances from the protostar;

—Weighted (Favre) averaging of the stochastic equations of motion of two-phase mechanics to phenomenologically describe the averaged gas-suspension flow and the processes of turbulent heat and mass transfer in a gas–dust disk;

—Deriving the defining relations for the correlation parameters of a turbulent two-phase flow needed to close the hydrodynamic equations of mean motion;

—Modeling the turbulent transfer coefficients in a gas–dust disk by taking into account the inverse effect of the polydisperse component on the turbulence intensity of the carrier gas.

## BASIC EQUATIONS OF THE MECHANICS OF HETEROGENEOUS MEDIA IN A PROTOPLANETARY GAS–DUST CLOUD

The motion of a gas suspension in a gas–dust accretion disk can probably be modeled most adequately in terms of the mechanics of heterogeneous turbulized media by taking into account the peculiarities of the physical–chemical properties of the phases, heat and mass transfer and radiation, chemical reactions, phase transitions, coagulation, fragmentation, etc. The evolution of such media is studied by invoking new thermohydrodynamic parameters and by solving more complex equations than those that we deal with in "ordinary" hydrodynamics. In this case, the intraphase and interphase interactions in heterogeneous media are occasionally very difficult to describe in detail, and rational schematizations leading to manageable and solvable equations are particularly needed here to obtain reliable results and to understand them.

In general, the continuity and energy equations and the equations of motion for each individual phase are used in the above and other known works on the continuum modeling of turbulized accretion disks with dust (Nakagawa *et al.*, 1981; Weidenschilling, 1984; Hayashi *et al.*, 1985; Nakagawa *et al.*, 1986; Schmitt *et al.*, 1997; Balbus and Hawley, 1998). The authors have to heuristically specify the laws of interphase interactions, in particular, the rates of momentum and energy transfer between the phases. This approach is an analog of Grad's thirteen-moment method (Grad, 1949), which gained wide acceptance, for example, in the kinetic theory of a multicomponent plasma. The subsequent averaging of the mutually coupled hydrodynamic equations for the individual phases (to describe the turbulized motions) leads not only to cumbersome equations of mean motion, which is related to the need for retaining a large number of correlation moments of the pulsating parameters (such as $\overline{\rho'_g \mathbf{u}'_g}$, $\overline{\rho'_d \mathbf{u}'_d \mathbf{u}'_d}$, $\overline{T' \mathbf{u}'_d}$, $\overline{\rho'_g T'}$, and the like) in their structure, but also to difficulties in physically interpreting each individual term in the averaged equations. To overcome these difficulties and to simplify the problem, some of the authors often unjustifiably discard a number of correlation terms, which, of course, narrows the range of application of such an approach.

At the same time, the evolution of a turbulized gas–dust cloud can be modeled in the single-velocity approximation of heterogeneous mechanics, which is similar to the Chapman–Enskog moment method of solving the system of Boltzmann kinetic equations for multicomponent gas mixtures (see, e.g., Chapman and Cowling, 1960; Marov and Kolesnichenko, 1987). In the case under consideration, this approach is peculiar in that, despite the hydrodynamic velocity difference between the individual phases and the related necessity of allowing for the dynamical and inertial effects of their relative motion, the continuum description of a disk medium can be made using the laws of conservation of mass, momentum, and energy for the system as a whole supplemented with the defining (closing) relations for a number of thermohydrodynamic flows, both intraphase and interphase ones. In particular, the generalized Stefan–Maxwell relations that were derived by Kolesnichenko and Maksimov (2001) for heterogeneous media with sufficient completeness and logical coherence by the methods of nonequilibrium thermodynamics can be used for the interphase diffusion flows (or the relative velocities of the phases). It is important to emphasize that using the total continuum alone to model the gas–dust cosmic material allows us, when using the Favre weighted averaging (Favre, 1969), to average the hydrodynamic equations for the disk medium as a whole fairly accurately (see, e.g., Marov and Kolesnichenko, 2002).

### *Basic Assumptions*

The huge variety, mutual influence, and complexity of the multiphase effects in the solar protoplanetary cloud (phase transitions, chemical reactions, heat transfer, gravitational interaction, pulsational and random motions, rotation, radiation, coagulation, etc.) necessarily require appropriately schematizing the description of the motion of a gas–dust medium. Therefore, in this paper, we will assume that the motion of a disperse mixture[9] in a protoplanetary disk can be adequately described under the following assumptions:

(1) the dust particles[10] (by which we mean below a solid-phase condensate) are solid, nondeformable, spherical in shape, and polydisperse;

(2) the dust particle material is assumed to be incompressible, $\rho_d = \text{const}$;

(3) the true dust density is much higher than the true gas density of the system, $\rho_d \gg \rho_g$;

(4) the volume concentration of the disperse phase is moderately high ($s^2 \ll 1$), so the terms of order $s^2$ may be neglected;

(5) the carrier phase is a compressible multicomponent perfect gas;

(6) the diffusion transfer of molecules of all chemical types relative to one another may be ignored, $\mathbf{u}_{\alpha(k)} \equiv \mathbf{u}_\alpha$;

(7) the viscosity and thermal conductivity of the disperse phase may be disregarded, $\mathbf{\Pi}_d = 0$ and $\mathbf{q}_d = 0$;

(8) the gas and disperse phases are assumed to be in thermal equilibrium, $T_g = T_d = T$;

(9) the total heterogeneous continuum is considered in the single-pressure approximation, $p_g = p_d = p(\rho_g, T)$;

(10) the heterogeneous reactions on the surfaces of solid particles may be disregarded;

(11) the contribution from the interphase boundaries[11] (the near-surface layer of solid particles) to the energetics of the disk system as a whole may be neglected;

(12) it is assumed that the rotation of the solid particles may be ignored when describing the dynamical interaction between the phases;

(13) the heat transfer between the disperse particles and the carrier gas may be disregarded.

Thus, we are going to model a heterogeneous continuum composed of two contacting phases, a carrier gas phase of solar composition and a disperse phase of solid condensed particles of complex chemical composition (see, e.g., Dorofeeva and Makalkin, 2004), at absolute temperature $T$ and pressure $p$[12] and well-mixed inside each macrovolume element $\delta\mathcal{V}$ of the disk medium.[13] Each phase is assumed to be a homogeneous $n$-component mixture (with each geochemically significant material of the protoplanetary cloud being present in each phase $\alpha$). Below, we use the Greek subscripts $\alpha, \beta, \ldots$ to denote the phases and refer the Latin subscripts in parentheses of type $(k)$ or $(i)$ for any quantity to the molecular component of the phase. The gas phase ($\alpha = g$) is the carrier medium[14] described by the model of a viscous fluid. The disperse phase ($\alpha = d$) present in the form of solid inclusions (however, we will not ignore the collisions between them) is inviscid and non-heat-conductive. For each of the two phases at each space-time point $(\mathbf{x}, t)$, we define the mass density, hydrodynamic velocity, internal energy, and other thermohydrodynamic parameters pertaining to its own continuum and its own chemical component of the mixture. As the phase parameters, we will use quantities averaged both over the total macrovolume element $\delta\mathcal{V} = \sum_\alpha \delta\mathcal{V}_\alpha$ pertaining to the heterogeneous system as a whole and over the part $\delta\mathcal{V}_\alpha$ of the volume element occupied by the individual phase $\alpha$. In particular, apart from the distributed (spread over the total volume $\delta\mathcal{V}$) mass density $\tilde{\rho}_\alpha$ of phase $\alpha$, we will use below the true (physical) density $\rho_\alpha$ (equal to the ratio of the mass of the phase-$\alpha$ particles in the macrovolume element $\delta\mathcal{V}$ to the part of this volume $\delta\mathcal{V}_\alpha$ occupied by the phase). The latter is defined by the expression

$$\rho_\alpha = \tilde{\rho}_\alpha / s_\alpha, \quad s_\alpha \equiv \delta\mathcal{V}_\alpha / \delta\mathcal{V}, \quad \sum_\alpha s_\alpha = 1, \qquad (1)$$

where $s_\alpha$ is the so-called volume content[15] (or volume concentration) of phase $\alpha$. The true ($\rho_\alpha$) rather than distributed ($\tilde{\rho}_\alpha$) phase density, together with other parameters of the state, such as the temperature $T_\alpha$, internal energy $e_\alpha$, and entropy $S_\alpha$, determines the thermodynamic properties of an elementary phase-$\alpha$ macroparticle in its various states. In addition, the values of $s_\alpha$ also directly affect the hydrodynamic motion of the phases, since it appears in the corresponding equations of motion. Concurrently, we will assume that $r$ independent chemical reactions, including the interphase reactions and the cases where the chemical transformations are reduced to the displacement of component $k$ from one phase to the other, are possible between the individual chemical components $k$ of the disk system.

Assuming a local thermodynamic equilibrium within each phase and a local thermal equilibrium between radiation and matter, we will use a phenome-

[9] From the standpoint of thermodynamics and mechanics of a continuum medium, the disperse phase may be treated as a "pseudogas" whose "pseudomolecules" are disperse particles.

[10] In this paper, by the dust particles we mean solid bodies with sizes from one micron to several hundred meters.

[11] The presence in heterogeneous systems of interphase boundaries modeled by mathematical surfaces on which the fields of various thermodynamic parameters become discontinuous severely complicates the continuum theory of multiphase multicomponent systems (see Nigmatulin, 1978).

[12] Note that this is the condition only for thermal and mechanical equilibrium of the phases, but not for total phase equilibrium, which additionally requires that the chemical potentials of the phases (which are the key concept of the theory of phase equilibrium) be equal. In addition, in chemical equilibrium, i.e., for an equilibrium distribution of chemical components between the two phases, their chemical potentials must have a constant value in both phases.

[13] For the continuum approximation to be applicable, the linear sizes of the macrovolume element $\delta\mathcal{V}$ must be much larger than those of the disperse inclusions, but much smaller than the hydrodynamic scale length $L_{hydr}$.

[14] Occasionally, we will use numbers instead of letters to denote the gas and condensed phases, referring the subscript $\alpha = 1$ to the gas phase and $\alpha = 2$ to the disperse-phase parameters.)

[15] The parameter $s_\alpha$ is often called phase saturation.

nological theory of multifluid interpenetrating continua to describe the hydrodynamic motions in a gas–dust medium (with the corresponding physical–chemical properties). In particular, this theory includes the dynamical effects due to the inequality of the hydrodynamic velocities $\mathbf{u}_\alpha$ of the phases in the system (see, e.g., Nigmatulin, 1978; Kolesnichenko and Maksimov, 2001).

*The Conservation of Mass*

**Monodisperse gas–dust medium.** Let us first consider the case where all condensed particles of the protoplanetary gas–dust disk in each macrovolume element $\delta\mathcal{V}$ have the same instantaneous hydrodynamic velocity $\mathbf{u}_d(\mathbf{x}, t)$, irrespective of their sizes. The mass density $\rho(\mathbf{x}, t)$ and the weighted mean hydrodynamic velocity $\mathbf{u}(\mathbf{x}, t)$ (the instantaneous center-of-mass velocity of the gas-suspension macrovolume element centered at point $\mathbf{x}$) of the gas–dust mixture as a whole are defined by

$$\rho = \sum_\alpha \rho_\alpha s_\alpha = \rho_g(1-s) + \rho_d s, \qquad (2)$$

$$\mathbf{u} = \rho^{-1}\sum_\alpha \rho_\alpha s_\alpha \mathbf{u}_\alpha = \frac{\rho_g(1-s)}{\rho}\mathbf{u}_g + \frac{\rho_d s}{\rho}\mathbf{u}_d, \qquad (3)$$

where $\rho_\alpha(\mathbf{x}, t)$ and $\mathbf{u}_\alpha(\mathbf{x}, t)$ are the true mass density and hydrodynamic velocity of phase $\alpha$, respectively; $s_d(\mathbf{x}, t)$ is the instantaneous volume concentration of the disperse phase, $(s_g + s_d = 1)$; below, we omit the subscript "d" of the parameter $s_d$, $s_d \equiv s$.

To model the chemical composition of a protoplanetary cloud, particularly at early stages of its evolution (see, e.g., Willacy *et al.*, 1998; Dorofeeva and Makalkin, 2004), we must generally invoke the mass balance equations for each chemical component of the phase, which can be represented as the equations for the conservation of particles of type $k$ in phase $\alpha$. Under the above assumptions, these equations take the form

$$\frac{\partial}{\partial t}(s_\alpha n_{\alpha(k)}) + \nabla(s_\alpha n_{\alpha(k)}\mathbf{u}_\alpha) = \sigma_{\alpha(k)} \equiv \sum_{\rho=1}^{r} \nu_{\alpha(k),\rho}\xi_\rho + \delta_{2\alpha}\overset{\circ}{n}_{\alpha(k)} \qquad (4)$$
$$(\alpha = 1, 2;\ k = 1, 2, \ldots, n)$$

or

$$\rho\frac{d}{dt}\left(\frac{s_\alpha n_{\alpha(k)}}{\rho}\right) + \nabla(s_\alpha n_{\alpha(k)}\mathbf{w}_\alpha) = \sigma_{\alpha(k)}, \quad \mathbf{w}_\alpha \equiv (\mathbf{u}_\alpha - \mathbf{u}). \qquad (5)$$

Here, $d(\ldots)/dt \equiv \partial(\ldots)/\partial t + \mathbf{u}\cdot\nabla(\ldots)$ is the substantial derivative related to the motion of the macrovolume element of the gas–dust medium as a whole; $\nabla(\ldots) = \sum_l \mathbf{i}_i\,\partial(\ldots)/\partial x_i$ is a vector differential operator; $\mathbf{i}_l$ $(l = 1, 2, 3)$ are the Cartesian unit vectors along the corresponding coordinate axes; the quantity $\nabla\cdot\mathbf{b}$ is the divergence of $\mathbf{b}$; $n_{\alpha(k)}(\mathbf{x}, t)$ is the number of particles of chemical substance $k$ in a unit volume occupied by phase $\alpha$ (countable concentration); $\mathbf{w}_\alpha(\mathbf{x}, t)$ is the instantaneous diffusion velocity of phase $\alpha$ that, by the definition of the weighted mean velocity $\mathbf{u}$, satisfies the relation

$$\sum_\alpha \rho_\alpha s_\alpha \mathbf{w}_\alpha = 0, \quad \mathbf{w}_\alpha \equiv (\mathbf{u}_\alpha - \mathbf{u}) \qquad (6)$$

or

$$\sum_\alpha \mathbf{J}_\alpha = 0, \quad \mathbf{J}_\alpha \equiv \rho_\alpha s_\alpha \mathbf{w}_\alpha = \rho_\alpha s_\alpha(\mathbf{u}_\alpha - \mathbf{u}), \qquad (7)$$

where $\rho_\alpha = \sum_k M_{(k)} n_{\alpha(k)}$; $\mathbf{J}_\alpha(\mathbf{x}, t)$ is the mass diffusion flux of phase-$\alpha$ particle; $\sigma_{\alpha(k)}$ is the production rate of particles of component $k$ per in a unit volume via chemical reactions and phase transitions (evaporation and condensation) as well as the fragmentation and coagulation of the disperse component; $\xi_\rho(\mathbf{x}, t)$ is the rate of chemical reaction $\rho$ (including the interphase reactions and phase transitions), $\rho = 1, 2, \ldots, r$; $\nu_{\alpha(k),\rho}$ is the stoichiometric coefficient of component $k$ in phase $\alpha$ with respect to chemical reaction $\rho$,[16] whose stoichiometric equation can be symbolically written as (see, e.g., Prigogine and Defay, 1966)

$$\sum_\alpha \sum_k \nu_{\alpha(k),\rho}\mathcal{M}_{(k)} = 0, \quad (\rho = 1, 2, \ldots, r), \qquad (8)$$

the principle of conservation of total mass in chemical reaction $\rho$; $\mathcal{M}_{(k)}$ is the molecular mass of component $k$; $\overset{\circ}{n}_{\alpha(k)}$ is a quantity that describes the change in the number density of chemical component $k$ in the dust phase related to the fragmentation or adhesion of condensed particles in the gas–dust cloud.

If the mass of all chemical components in the dust phase is conserved during the transformation of solid particles ($\sum_k \mathcal{M}_{(k)}\overset{\circ}{n}_{\alpha(k)} = 0$), then the following differential conservation equation follows from Eq. (4):

$$\rho\frac{d}{dt}\left(\frac{\rho_\alpha s_\alpha}{\rho}\right) + \nabla(\rho_\alpha s_\alpha \mathbf{w}_\alpha) = \sigma_{\alpha\beta} = \sum_{\rho=1}^{r}\nu_{\alpha,\rho}\xi_\rho \qquad (9)$$
$$(\alpha, \beta = 1, 2)$$

[16] We assume that the stoichiometric coefficients of the components produced during the reaction (from left to right) are positive, while the coefficients of the expendable components are negative.

for the distributed mass density

$$\tilde{\rho}_\alpha \equiv s_\alpha \sum_k \mathcal{M}_{(k)} n_{\alpha(k)} = \rho_\alpha s_\alpha \tag{10}$$

of phase α. Here, $\nu_{\alpha,\rho} \equiv \sum_k \mathcal{M}_{(k)} \nu_{\alpha(k),\rho}$; the quantity $\sigma_{\alpha\beta}(\mathbf{x}, t)$ characterizes the rate of mass transformation from phase α to phase β (or vice versa, then $\sigma_{\alpha\beta} < 0$) via chemical reactions and the evaporation or condensation of material in the protoplanetary cloud; in this case, $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$. For our subsequent purposes, it is appropriate to introduce the mass phase concentrations $C_\alpha(\mathbf{x}, t)$,

$$C_\alpha \equiv \frac{\tilde{\rho}_\alpha}{\rho} = \frac{\rho_\alpha s_\alpha}{\rho}, \quad \sum_\alpha C_\alpha = 1, \tag{11}$$

and the relative velocity of the dust and gas, $\mathbf{w} \equiv \mathbf{w}_{dg} = (\mathbf{u}_d - \mathbf{u}_g)$; Eqs. (9) can then be written in a more compact form,

$$\rho \frac{dC_\alpha}{dt} = -\nabla \cdot \mathbf{J}_\alpha + \sigma_{\alpha\beta} \quad (\alpha, \beta = 1, 2),$$
$$\mathbf{J}_{1,2} = \begin{cases} \rho C_1 \mathbf{w}_1 = -\rho C_1 C_2 \mathbf{w} \\ \rho C_2 \mathbf{w}_2 = \rho C_1 C_2 \mathbf{w}. \end{cases} \tag{12}$$

Given (7) and (8), the law of conservation of total mass obtained by summing (4) over the indices $k$ and α takes a standard form,

$$\rho \frac{d}{dt}\left(\frac{1}{\rho}\right) - \nabla \cdot \mathbf{u} = 0, \text{ or } \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{13}$$

as in a single-phase continuum. Note that radiation does not modify the continuity equation (13), since it “does not possess” any mass.

Next, we will assume that the material of the solid inclusions remains incompressible during the evolution of the gas–dust cloud, i.e., the true (physical) dust density is $\rho_d$ = const. The instantaneous equation (12) for dust is then be reduced to the equation

$$\rho \frac{d}{dt}\left(\frac{s}{\rho}\right) = -\nabla \cdot (s \mathbf{w}_d) + \rho_d^{-1} \sigma_{dg},$$
$$\mathbf{w}_d = C_g \mathbf{w} = (1 - s) \frac{\rho_g}{\rho} \mathbf{w}, \tag{14}$$

which allows the volume content $s(\mathbf{x}, t)$ of the dust component in a two-phase flow to be found at a given relative velocity of the phases **w**. Thus, Eqs. (13) and (14) can be used instead of the two equations (9) to calculate the parameters ρ and $s$ (and, hence, the gas density $\rho_g$ [see Eq. (2)].

The intensity of the force interaction between the phases and the radiation parameters in a gas–dust cloud strongly depend on the characteristic size of the solid inclusions (e.g., the characteristic volume of one dust particle $\tilde{U}_d(\mathbf{x}, t)$) and their total number $N_d \equiv s \sum_k n_{d(k)}$ in a unit total gas-suspension volume. If all solid-phase condensates are spherical or nearly spherical with the characteristic Feret diameter $\tilde{d}(\mathbf{x}, t)$, then $\tilde{U}_d = (\pi/6)\tilde{d}^3$. The balance equation for the total number of disperse particles $N_d(\mathbf{x}, t)$ can be derived from (4):

$$\rho \frac{d}{dt}\left(\frac{N_d}{\rho}\right) + \nabla \cdot (N_d \mathbf{w}_d) = \sigma_{N_d}$$
$$\equiv \sum_k \sum_{\rho=1}^{r} \nu_{d(k),\rho} \xi_\rho + \overset{\circ}{N}_d, \tag{15}$$

where the source term $\overset{\circ}{N}_d \equiv \sum_k \overset{\circ}{n}_{\alpha(k)}$, which characterizes the change in the total number density of different-scale solid particles via coagulation and fragmentation, is generally defined by the Smoluchowski kinetic equation (see Eq. (29)). An important parameter of the coagulating mixture in a two-phase flow, the characteristic volume $\tilde{U}_d(\mathbf{x}, t)$ (or the mean linear diameter $\tilde{d}_d$) of the solid inclusions, can be determined from the known parameters $N_d$ and $s$:

$$\tilde{U}_d = s/N_d, \quad \tilde{d}_d = \sqrt[3]{(6/\pi)(s/N_d)}. \tag{16}$$

If we disregard the fragmentation and coagulation of solid particles ($\overset{\circ}{N}_d = 0$) and the evaporation and condensation processes ($\sigma_{dg} = 0$) in our numerical simulations of the evolution of a gas–dust cloud and assume that the material of the inclusions is incompressible ($\rho_d$ = const), then we will have $\tilde{U}_d$ = const. Under these assumptions, $\tilde{\rho}_d = s\rho_d = \rho_d \tilde{U}_d N_d$, and Eq. (15) (with a zero right-hand side) is just a corollary of the mass conservation equation (14) for the second phase, which in this case takes a simple form,

$$\rho \frac{d}{dt}\left(\frac{s}{\rho}\right) + \nabla \cdot (s \mathbf{w}_d) = 0, \quad \mathbf{w}_d = C_g \mathbf{w}. \tag{14*}$$

### *Interphase Diffusion*

As we already said above, the generalized Stefan–Maxwell relations can serve as the basic equations in analyzing the phase diffusion in a gas–dust cloud. Using the kinetic theory, Hirschfelder, Curtiss, and Bird (1954) considered in detail the diffusion in a multicomponent mixture of gases in their monograph. Their main result is that the relative diffusion velocity $(\mathbf{u}_{(j)} - \mathbf{u}_{(k)})$ of two gas components can be caused by factors that do not directly affect these components, for example, by the thermodynamic forces acting on the molecules of the other mixture components. The velocities $(\mathbf{u}_{(j)} - \mathbf{u}_{(k)})$ can be found (to the first approximation of the Chapman–Enskog method of solving the Boltzmann equations) from the system of Stefan–Maxwell equations

$$\sum_j \frac{n_{(j)}n_{(k)}}{n^2\mathcal{D}_{(jk)}}(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}) - k_{T(k)}\nabla \ln T = \mathbf{d}_{(k)}$$

$$\equiv \frac{1}{p}\left\{-\rho_{(k)}\mathbf{F}_{(k)} + \nabla p_{(k)} - C_{(k)}\sum_j(-\rho_{(j)}\mathbf{F}_{(j)} + \nabla p_{(j)})\right\}$$

$$(k = 1, 2, \dots),$$

which can be written in the form of equations of motion for the individual mixture components (see, e.g., Note I in the monograph by Chapman and Cowling (1960)),

$$\rho_{(k)}\frac{d\mathbf{u}}{dt} = -\nabla p_{(k)} + \rho_{(k)}\mathbf{F}_{(k)}$$

$$+\sum_j k_B T\frac{n_{(j)}n_{(k)}}{n\mathcal{D}_{(jk)}}(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}) - k_B k_{T(k)} n\nabla T.$$

Here, $\rho_{(k)} = M_{(k)}n_{(k)}$, $C_{(k)} = \rho_{(k)}/\rho$, and $p_{(k)}$ are the mass density, mass concentration, and partial pressure of the particles of type $k$, respectively; $p = \sum_k p_{(k)}$ is the total pressure of the mixture (the Dalton law), $p_{(k)} = n_{(k)}k_B T$; $k_B$ is the Boltzmann constant; $n = \sum_k n_{(k)}$, and $\rho = \sum_k \rho_{(k)}$ are the total number and mass densities of the multicomponent mixture, respectively; $\mathcal{D}_{(jk)}$ are the binary diffusion coefficients; $\mathbf{F}_{(k)}$ is the external bulk force (per unit mass of component $k$); and $k_{T(k)}$ is the thermodiffusion ratio.

Kolesnichenko (1998) derived these Stefan–Maxwell relations for multicomponent mixtures by the methods of nonequilibrium thermodynamics, while Kolesnichenko and Maksimov (2001) generalized them to heterogeneous mixtures (they can also be written in the form of equations of motion for the individual phases of the system). In the single-pressure approximation ($p_g = p_d$) considered here, these relations for the interphase diffusion take the form

$$\sum_\beta R_{\alpha\beta}(\mathbf{u}_\beta - \mathbf{u}_\alpha) - pk_{T\alpha}\frac{\nabla T}{T}$$

$$= \mathbf{d}_\alpha \equiv -\tilde{\rho}_\alpha\mathbf{K}_\alpha + s_\alpha\nabla p_\alpha - C_\alpha\sum_\beta(-\tilde{\rho}_\beta\mathbf{K}_\beta + s_\beta\nabla p_\beta) \quad (17)$$

$$\equiv \rho_\alpha s_\alpha\frac{d_\alpha\mathbf{u}_\alpha}{dt} + s_\alpha\nabla p - \rho_\alpha s_\alpha\mathbf{F}_\alpha - \nabla\cdot\mathbf{\Pi}_\alpha$$

$$+\frac{1}{2}\sum_\beta\sigma_{\alpha\beta}(\mathbf{u}_\alpha - \mathbf{u}_\beta) \quad (\alpha = 1, 2, \dots),$$

where $d_\alpha(\dots)/dt = \partial(\dots)/\partial t + \mathbf{u}_\alpha\cdot\nabla(\dots)$ is the substantial derivative along the trajectory of the center of mass of phase $\alpha$ contained inside the macrovolume element $\delta\mathcal{V}$ of a multiphase medium; $\mathbf{\Pi}_\alpha$ is the partial viscous stress tensor; $R_{\alpha\beta}$ is the coefficient of interphase friction for phases $\alpha$ and $\beta$ (since the coefficient $R_{\alpha\beta}$ reflects the interaction between two phase continua, it is often convenient to write in a symmetric form, $R_{\alpha\beta} = \tilde{\rho}_\alpha\tilde{\rho}_\beta\theta_{\alpha\beta}$, where the parameters $\theta_{\alpha\beta}$ do not depend, at least roughly, on the mixture proportions), $R_{\alpha\beta} = R_{\beta\alpha}$;[17] $\mathbf{F}_\alpha$ is the external bulk force per unit mass of phase $\alpha$;

$$\tilde{\rho}_\alpha\mathbf{K}_\alpha \equiv -\tilde{\rho}_\alpha\frac{d_\alpha\mathbf{u}_\alpha}{dt} + \tilde{\rho}_\alpha\mathbf{F}_\alpha + \nabla\cdot\mathbf{\Pi}_\alpha - \frac{1}{2}\sum_\beta\sigma_{\alpha\beta}\mathbf{w}_{\alpha\beta} \quad (18)$$

is the generalized thermodynamic force coupled with the diffusion flux $\mathbf{J}_\alpha$; the quantities $\mathbf{d}_\alpha$ also have the meaning of generalized thermodynamic forces that cause the relative motion of the phases ($\sum_\alpha \mathbf{d}_\alpha = 0$); $k_{T\alpha}$ is the thermophoretic ratio, with $\sum_\alpha k_{T\alpha} = 0$. A number of theoretical formulas were suggested in the literature (see, e.g., Soo *et al.*, 1960; Mednikov, 1981) to determine the thermophoretic force. It should be noted, however, that the thermophoresis-related force may be disregarded for disperse particles whose sizes are generally much smaller than the temperature nonuniformity scale length in the disk. The last term on the right-hand side of Eq. (17), $\frac{1}{2}\sum_\beta \mathbf{w}_{\alpha\beta}\sum_{\rho=1}^{r}\nu_{\alpha,\rho}\xi_\rho$, describes the change in the momentum of phase $\alpha$ via phase transitions (recall that $\mathbf{w}_{\alpha\beta} = \mathbf{u}_\alpha - \mathbf{u}_\beta$). In the case of a two-phase gas–dust disk system under consideration, this term take into account the loss of momentum by the dust-phase particles through the mass transformation of some of them to the gas phase during evaporation or the acquisition of additional momentum by the disperse phase through the formation (from the gas) of new solid particles during condensation. Nevertheless, below we will also disregard this term in most cases, since it is almost always much smaller than the Stokes force of friction, $\mathbf{F}_{\text{fric},\alpha} = \sum_\beta R_{\alpha\beta}\mathbf{w}_{\alpha\beta}$ arising from phase viscosity effects (see, e.g., Nigmatulin, 1987) and is exactly equal to zero in the absence of phase transitions. It is important to emphasize once again that, in contrast to the classical inertia-free Stefan–Maxwell relations for the relative velocities of the components

[17] A more detailed analysis of the interphase interaction shows that large gradients in macroscopic parameters, the rotation of solid particles, nonstationary establishment of the velocity profile near the particles, the deformation of disperse particles, and some other effects can give rise to additional forces on the left-hand side of Eqs. (17). Apart from the Stokes force of friction, the Safman force (related to the nonuniformity of the carrier-gas velocity profile, the "hereditary" Basset force (which takes into account the prehistory of the motion on the behavior of disperse particles), and the Magnus or Zhukowski force (the force of additional action on rotating disperse particles due to the gradients in the mean velocity field of the carrier phase), etc. can be such forces.

$\mathbf{w}_{(jk)} = \mathbf{u}_{(j)} - \mathbf{u}_{(k)}$, the interphase diffusion laws described by the generalized equations (17) incorporate the inertia of the relative phase motion.

For a two-phase gas–dust disk medium, Eqs. (17) take the form of equations of motion for gas and dust[18]

$$\begin{cases} R_{\rm gd}\mathbf{w} \equiv (1-s)s\rho_{\rm d}\rho_{\rm g}\theta_{\rm dg}\mathbf{w} = \rho_{\rm g}(1-s)\dfrac{{\rm d_g}\mathbf{u}_{\rm g}}{{\rm d}t} \\ + (1-s)\nabla p - \rho_{\rm g}(1-s)\mathbf{g} - \nabla\cdot\mathbf{\Pi}_{\rm g} \\ -R_{\rm dg}\mathbf{w} = \rho_{\rm d}s\dfrac{{\rm d_d}\mathbf{u}_{\rm d}}{{\rm d}t} + s\nabla p - \rho_{\rm d}s\mathbf{g}, \end{cases} \tag{17*}$$

where $\mathbf{F}_{\rm d} = \mathbf{F}_{\rm g} \equiv \mathbf{g}(\mathbf{x}, t)$ is the bulk force per unit mass, which is generally related to both the gravitational attraction by the star and the gravitational attraction by the gas–dust cloud itself.

The sought-for diffusion relation for the relative velocity vector of the dust and the gas $\mathbf{w} \equiv (\mathbf{u}_{\rm d} - \mathbf{u}_{\rm g})$ can be derived from the terms of Eqs. (17*) that describe the action of friction if we divide each of them by the corresponding quantity $s_\alpha\rho_\alpha$, subtract one from the other, and separate out the term with $\mathbf{w}$. Writing the true velocities of the phases as $\mathbf{u}_{\rm d} = (C_{\rm g}\mathbf{w} + \mathbf{u})$ and $\mathbf{u}_{\rm g} = (-C_{\rm d}\mathbf{w} + \mathbf{u})$ and assuming that $\rho_{\rm d} \gg \rho_{\rm g}$, we obtain a defining relation for $\mathbf{w}$ (an analog of the Darcy law for filtration),

$$\frac{\rho}{\tilde\rho_{\rm d}\tilde\rho_{\rm g}}R_{\rm dg}\mathbf{w} \equiv \rho\theta_{\rm gd}\mathbf{w} = \frac{{\rm d_g}\mathbf{u}_{\rm g}}{{\rm d}t} - \frac{{\rm d_d}\mathbf{u}_{\rm d}}{{\rm d}t} + \frac{\rho_{\rm d}-\rho_{\rm g}}{\rho_{\rm g}\rho_{\rm d}}\nabla p - \frac{1}{\tilde\rho_{\rm g}}\nabla\cdot\mathbf{\Pi}_{\rm g} \cong -\frac{{\rm d}\mathbf{w}}{{\rm d}t} + \frac{1}{\rho_{\rm g}}\nabla p, \tag{19}$$

which is below considered as the main equation in modeling the phase diffusion in the disk. Note that the gravitational forces after this subtraction canceled out, but their action on the motion of the gas–dust medium manifests itself through a pressure gradient (see, e.g., Eqs. (192) and (194)). In writing (19), we made no distinction between the substantial derivatives for the individual phases and the system as a whole; i.e., we assumed that ${\rm d_d}/{\rm d}t \cong {\rm d_g}/{\rm d}t \cong {\rm d}/{\rm d}t$, which is valid in the mechanics of mixtures only in the so-called diffusion approximation. In general, i.e., when the acceleration of the diffusion flows relative to the center of mass is taken into account, the following exact transformation should be used:

$$\begin{aligned} \frac{{\rm d_d}\mathbf{u}_{\rm d}}{{\rm d}t} - \frac{{\rm d_g}\mathbf{u}_{\rm g}}{{\rm d}t} &= \frac{{\rm d}\mathbf{w}}{{\rm d}t} + (\mathbf{w}_{\rm d}\cdot\nabla)\mathbf{u}_{\rm d} - (\mathbf{w}_{\rm g}\cdot\nabla)\mathbf{u}_{\rm g} \\ &= \frac{{\rm d}\mathbf{w}}{{\rm d}t} + (\mathbf{w}\cdot\nabla)\mathbf{u} + C_{\rm g}(\mathbf{w}\cdot\nabla)C_{\rm g}\mathbf{w} - C_{\rm d}(\mathbf{w}\cdot\nabla)C_{\rm d}\mathbf{w} \\ &= \frac{{\rm d}\mathbf{w}}{{\rm d}t} + (\mathbf{w}\cdot\nabla)\mathbf{u} + \mathcal{J}(\mathbf{w}^2) \approx \frac{{\rm d}\mathbf{w}}{{\rm d}t} + (\mathbf{w}\cdot\nabla)\mathbf{u}, \end{aligned} \tag{20}$$

where $\mathcal{J}(\mathbf{w}^2) \equiv C_{\rm g}(\mathbf{w}\cdot\nabla)C_{\rm g}\mathbf{w} - C_{\rm d}(\mathbf{w}\cdot\nabla)C_{\rm d}\mathbf{w}$ is a quadratic (in $\mathbf{w}$) function; the latter may often be omitted (see Youdin and Goodman, 2004), in particular, for a fine passive admixture, since $|\mathbf{w}|$ is small for them. The defining diffusion relation for the relative phase velocity vector then takes a more complex form:

$$\rho\theta_{\rm gd}\mathbf{w} \cong -\frac{{\rm d}\mathbf{w}}{{\rm d}t} - (\mathbf{w}\cdot\nabla)\mathbf{u} - C_{\rm g}(\mathbf{w}\cdot\nabla)C_{\rm g}\mathbf{w} + C_{\rm d}(\mathbf{w}\cdot\nabla)C_{\rm d}\mathbf{w} + \frac{\nabla p}{\rho_{\rm g}}. \tag{19*}$$

**The coefficient of resistance.** The coefficient of friction $R_{\rm dg}$ between gas and dust continua is defined in the literature by various formulas, depending on the characteristic diameter $\tilde d_{\rm d}$ of the disperse-phase particles (see, e.g., Sternin *et al.*, 1980; Shraiber *et al.*, 1987). If the characteristic diameter of spherical solid particles $\tilde d_{\rm d} \ll \lambda_{\rm g}$, where $\lambda_{\rm g}$ is the mean free path of molecules in the gas phase, then $R_{\rm dg}$ is given by the Epstein formula (Epstein, 1924). For course spherical condensates with diameters larger than the mean free path of gas molecules, the coefficient of resistance is defined by the Stokes law (Stokes, 1851). Thus, for the coefficients of resistance $R_{\rm dg}$ (or $\theta_{\rm dg}$) of a smooth spherical particle, we have (see, e.g., Weidenschilling, 1977; Garaud *et al.*, 2005),

$$R_{\rm dg} = \begin{cases} \dfrac{2\tilde\rho_{\rm d}\tilde\rho_{\rm g}c_{\rm sg}}{\tilde d_{\rm d}\rho_{\rm d}}, \\ \text{when } \tilde d_{\rm d} \ll \lambda_{\rm g} \text{ (Epstein regime)} \\ \dfrac{2\tilde\rho_{\rm d}\tilde\rho_{\rm g}}{\tilde d_{\rm d}\rho_{\rm d}}C_{\rm D}({\rm Re_d})|\mathbf{w}|, \\ \text{when } \tilde d_{\rm d} \gg \lambda_{\rm g} \text{ (Stokes regime)}, \end{cases} \tag{21}$$

[18] Note that the general forms of the equations of motion and the continuity equations in our paper and those of various authors who studied the gas suspensions in accretion disks are identical. The only difference is the presence of the term $-s\nabla p$ in the equation of motion (17*) for gas and the absence of the term $s\nabla p$ in the equation of motion (17*) for dust. Their appearance in continuum heterogeneous mechanics is eventually related to allowance for the effect of associated masses (due to the accelerated motion of the solid particles relative to the carrier gas, when disturbances of the order of the particle size emerge in the latter) and the buoyant force, which are often much smaller than the other terms in the equations of motion at large $\rho_d/\rho_g$ (typical of gas flows with solid particles) (see Nigmatulin, 1978). At the same time, since these forces are proportional not only to the gas density, but also to the local acceleration of the medium or the difference between the local accelerations of the gas medium and the solid particles, the situations where these additional terms of Eqs. (17*) will be comparable to the aerodynamic Stokes force are quite possible.

or

$$\theta_{dg} = \begin{cases} \dfrac{2c_{sg}}{\tilde{d}_d\rho_d}, & \text{when } \tilde{d}_d \ll \lambda_g \\ \dfrac{2}{\tilde{d}_d\rho_d} C_D(\mathrm{Re}_d)|\mathbf{w}|, & \text{when } \tilde{d}_d \gg \lambda_g. \end{cases} \quad (21^*)$$

Here, $c_{sg}$ is the speed of sound in the gas (see Eq. (57) below); $\mathrm{Re}_d = \tilde{d}_d|\mathbf{w}|/\nu_g$ is the Reynolds number for the dust; $\nu_g$ is the coefficient of molecular kinematic viscosity for the gas component of the mixture, $\nu_g = \lambda_g c_g/2$; $C_D(\mathrm{Re}_d)$ is the coefficient of aerodynamic resistance (the so-called standard resistance curve), which has a fairly complex form. A considerable number of formulas that fit this curve are known in the literature (see, e.g., Schlichting, 1974; Sternin *et al.*, 1980; Mednikov, 1981). In particular, the following expression gained wide acceptance in astrophysics (Whipple, 1972):

$$C_D(\mathrm{Re}_d) = \begin{cases} 9\mathrm{Re}_d^{-1}, & \mathrm{Re}_d \le 1 \\ 9\mathrm{Re}_d^{-0.6}, & 1 \le \mathrm{Re}_d \le 800 \\ 0.165, & \mathrm{Re}_d \ge 800. \end{cases} \quad (22)$$

In our view, the following trinomial formula is no less convenient:

$$C_D(\mathrm{Re}_d) = 9\mathrm{Re}_d^{-1}(1 + 0.179\mathrm{Re}_d^{1/2} + 0.013\mathrm{Re}_d), \quad (0.1 < \mathrm{Re}_d < 10^3); \quad (22^*)$$

its advantage is that it is applicable over a wide $\mathrm{Re}_d$ range.

It should be noted that, in general, the conditions for the flow around particles in actual multiphase flows differ significantly from the idealized conditions in which the standard curve is applicable. The particles in a gas–dust cloud generally have an irregular shape and a rough surface and move nonuniformly in a turbulized flow of rarefied and compressible gas. Of course, each of these factors changes (sometimes significantly) the conditions for the flow around a particle in a disk and the force of aerodynamic resistance. Let us briefly consider their effect that, as a rule, is disregarded in astrophysical literature.

(1) It is customary to characterize the degree of deviation of the particle shape from a sphere in heterogeneous mechanics by the shape factor $\beta$ ($\beta \ge 1$), the ratio of the surface area of an actual particle to the surface area of a sphere of the same volume. Gorbis (1970) suggested formulas for calculating the coefficients of aerodynamic resistance $C_D(\mathrm{Re}_d, \beta)$, which have higher values than those of the standard curve for essentially nonisometric dust particle shapes.

(2) Sternin *et al.* (1980) established that the coefficient of resistance $C_D(\mathrm{Re}_d)$ for particles with appreciable roughness increases (compared to the standard curve) if the latter is comparable to the thickness of the boundary layer.

(3) Flow turbulization also affects significantly $C_D(\mathrm{Re}_d)$. As was pointed out by Sternin and Shraiber (1994), according to the data of various authors, for example, for $20 < \mathrm{Re}_d < 100$, $C_D$ varies within the range $(0.01–3)C_D^*$ (in what follows, $C_D^*$ corresponds to the standard curve). It is important to note that the effect of turbulence decreases with decreasing Reynolds number $\mathrm{Re}_d$. For comparatively small $\mathrm{Re}_d$, the Lopez–Dackler formulas (see, e.g., Sternin *et al.*, 1980) can be used:

$$C_D(\mathrm{Re}_d) = \begin{cases} 60.75\varepsilon^{1/3}\mathrm{Re}_d^{-1}, \\ \mathrm{Re}_d < 50, \; 0.05 < \varepsilon < 0.5 \\ 0.0498(1 + 150/\mathrm{Re}_d)^{1.565} + 1.5\varepsilon, \\ 50 < \mathrm{Re}_d < \mathrm{Re}_d^*, \quad 0.07 < \varepsilon < 0.5, \end{cases}$$

where $\varepsilon$ is the relative turbulence level, i.e., the ratio of the root-mean-square pulsation velocity to the averaged sliding velocity; $\mathrm{Re}_d^* = \min\{0.9\mathrm{Re}_{crit}, 700\}$; $\mathrm{Re}_{crit}$ is the critical Reynolds number; $\ln\mathrm{Re}_{crit} = 5.477–15.8\varepsilon$ ($\varepsilon \le 0.15$); $\ln\mathrm{Re}_{crit} = 3.371–1.75\varepsilon$ ($\varepsilon > 0.15$).

(4) The onflow compressibility and rarefaction affect significantly the aerodynamic resistance of the particles. The role of these factors is determined primarily by the Mach, $\mathrm{Ma} = |\mathbf{u}_g|/c_{sg}$, and Knudsen, Kn, numbers. The compressibility of the carrier gas plays a significant role in a high-velocity flow of gas suspension in a disk. From the numerous generalized relations available in the literature, we give the following formula (see, e.g., Sternin *et al.*, 1980):

$$C_D(\mathrm{Re}_d) = 9\mathrm{Re}_d^{-1}(1 + 0.179\mathrm{Re}_d^{1/2} + 0.013\mathrm{Re}_d) \times \frac{[1 + \exp(-0.427\mathrm{Ma}^{-4.63} - 3\mathrm{Re}_d^{-0.88})]}{1 + \mathrm{Re}_d^{-1}\mathrm{Ma}[3.82 + 1.28\exp(-1.25\mathrm{Ma}^{-1}\mathrm{Re}_d)]}, \quad (22^{**})$$

where $\mathrm{Re}_d < 100$ and $\mathrm{Ma} < 2$. Here, the first two factors in the numerator correspond to the standard curve, the third factor takes into account the compressibility effect, and the denominator take into account the rarefaction effect.

Hunter *et al.* (1981) gave the following relation for $\mathrm{Re}_d < 10^3$:

$$C_D(\mathrm{Re}_d) = (C_D^* - 2) \times \exp\left[-3.07\gamma^{1/2}\frac{\mathrm{Ma}}{\mathrm{Re}_d}\frac{1 + \mathrm{Re}_d(12/28 + 0.584\mathrm{Re}_d)}{1 + 11.28\mathrm{Re}_d}\right] + \frac{1}{\gamma^{1/2}\mathrm{Ma}}\left[\frac{5.6}{\mathrm{Ma}+1} + 1.7\left(\frac{T_d}{T}\right)^{1/2}\right]\exp\left(-\frac{\mathrm{Re}_d}{2\mathrm{Ma}}\right) + 2,$$

where $T_d$ is the dust particle temperature.

(5) At a reduced gas pressure[19] in the disk, the gas component of the medium can slide over the surface of a solid particle, which also causes the coefficient of aerodynamic resistance to decrease. The rarefaction of the medium is characterized by the Knudsen number, $\mathrm{Kn} = \lambda_g/\tilde{d}_d$. Four Kn ranges are usually distinguished: Knudsen flow (Kn > 10), transition regime (10 > Kn > 0.25), sliding flow (0.25 > Kn > 0.01), and continuum flow (there is no rarefaction effect, Kn < 0.01). For the first three ranges, the coefficient of aerodynamic resistance may be represented as $C_D = \varphi C_D^*$, where the coefficient $\varphi$ is defined by the well-known Millikan formula (see, e.g., Fuks, 1955)

$$\varphi = \{1 + \mathrm{Kn}[1.55 + 0.471\exp(-0.596/\mathrm{Kn})]\}^{-1}.$$

As the rarefaction increases, the effect of compressibility on the coefficient of resistance degenerates. All the above improvements of formula (22) can be easily taken into account when numerically simulating the structure of a protoplanetary gas–dust disk, for example, at the subdisk formation stage.

Returning to Eq. (21*) for the coefficient $\theta_{dg}(\mathrm{Re}_d)$, note that Eqs. (21*) are convenient only for monodisperse dust with a given characteristic linear diameter of the inclusions $\tilde{d}_d$, since in this case $\theta_{dg}$ does not depend on the volume concentration of the disperse phase $s$ and the total number density of the solid particles $N_d$. However, when the coagulation processes in the gas–dust protoplanetary cloud are taken into account, i.e., given that the dust is multifractional, it is appropriate to rewrite (21*) in a form that explicitly depends on the parameters $s$ and $N_d$, which are defined by the Smoluchowski equation. When using formula (16), Eq. (21*) for $\theta_{dg}(s, N_d, \mathrm{Re}_d)$ transforms to

$$\theta_{dg} = \theta_{dg}(s, N_d, \mathrm{Re}_d) = \begin{cases} (4/3\pi)^{1/3}\rho_d^{-1}s^{-1/3}c_{sg}N_d^{1/3}, & \text{when } \tilde{d}_d \ll \lambda_g \\ (4/3\pi)^{1/3}\rho_d^{-1}s^{-1/3}N_d^{1/3}C_D(\mathrm{Re}_d)|\mathbf{w}|, & \text{when } \tilde{d}_d \gg \lambda_g. \end{cases} \quad (23)$$

*Allowance for Dust Multifractionality*

Let us now consider in more detail the technique of calculating $N_d(\mathbf{x}, t)$ when the dust multifractionality of the system is taken into account. The actual protoplanetary cloud is polydisperse; i.e., condensed particles of various sizes $d_{d,k}$ are present in the macrovolume element $\delta\mathcal{V}$. This factor can be taken into account by breaking down the dust component into a finite number of fractions each of which is generally characterized by its own thermohydrodynamic parameters; i.e., instead of one disperse phase, we must consider $m$ phases (where $m$ is the number of fractions) each of which has its own macroparameters,

$$d_{d,k},\ n_{d,k},\ s_{d,k} = n_{d,k}(\pi/6)d_{d,k}^3, \quad \rho_{d,k},\ \mathbf{u}_{d,k}\ldots \quad (k = 1, \ldots, m), \quad (24)$$

where $\mathbf{u}_{d,k}(\mathbf{x}, t)$ is the hydrodynamic velocity of the fraction-$k$ solid particles.

Next, we assume that the material of different fractions is the same ($\rho_{d,1} = \rho_{d,2} = \ldots = \rho_{d,m} = \rho_d = \mathrm{const}$) and that the solid-phase condensates of fraction 1 constitute the group of the smallest (primary) particles, those of fraction 2 constitute the group of double particles, etc. up to the maximum size. To simplify our analysis of the coagulation process in an $(m + 1)$-phase polydisperse flow, we also assume that all solid particles are spherical or nearly spherical with the Feret diameter $d_{d,k}$. Since the size of chemically identical solid particles after their adhesion increases as the cubic root of the number of its constituent primary condensates ($d_{d,k} = d_{d,1}\sqrt[3]{k}$), the volume concentration of the fraction-$k$ disperse particles is defined by the relation

$$s_{d,k} = n_{d,k}(\pi/6)d_{d,k}^3 = U_1 k n_{d,k}, \quad (25)$$

where $n_{d,k}(\mathbf{x}, t)$ is the number density of the fraction-$k$ particles (their number in a unit total gas-suspension volume); $U_1 = (\pi/6)d^3$ and $d \equiv d_{d,1}$ are, respectively, the volume and diameter of one smallest particle. The volume content $s(\mathbf{x}, t)$, the distributed mass density $\tilde{\rho}_d(\mathbf{x}, t)$, and the hydrodynamic velocity $\mathbf{u}_d(\mathbf{x}, t)$ of the entire dust continuum can then be expressed as

$$s \equiv \sum_{k=1}^{m} s_{d,k} = U_1 \sum_{k=1}^{m} k n_{d,k}, \quad \tilde{\rho}_d = \rho_d \sum_{k=1}^{m} s_{d,k}, \quad s\mathbf{u}_d = \sum_{k=1}^{m} s_{d,k}\mathbf{u}_{d,k}. \quad (26)$$

In a disperse mixture in which the macroscopic velocities of the fractions differ, i.e., fractions $j$ and $k$ move relative to one another with the velocity $\mathbf{u}_{d,j} - \mathbf{u}_{d,k}$ ($j, k = 1, \ldots, m$), there will be collisions between particles of different fractions, which will lead to the mass, momentum, and energy transfer between the fractions. Allowance for this fact, which is important at the final subdisk formation stage (when the fraction of particles "reflected" from the subdisk with a mean or macroscopic velocity different from that of the "incident" particles can appear in the flow) and at the planetesimal formation stage (after the subdisk disintegration), severely complicates the problem of modeling the evolution of a protoplanetary gas–dust disk (see

[19]For example, Wasson (1985) obtained the following pressure estimates in the midplane of the circumsolar disk: $2 \times 10^{-5}$–$10^{-1}$ g/cm$^3$ at $r = 1$ AU and $5 \times 10^{-7}$–$2 \times 10^{-6}$ g/cm$^3$ at $r = 3$ AU.

Kolesnichenko, 2001). In this paper, however, we assume that the particles of the material ("pseudomolecules") belonging to different dust continua (fractions) move with the same hydrodynamic velocity, $\mathbf{u}_{d,k} \equiv \mathbf{u}_d$ $(k = 1, \ldots, m)$.

We have already mentioned above that the fragmentation and coagulation processes are the main size formation mechanisms of large solid particles as they accumulate in a gas–dust cloud. The fragmentation mechanism of colliding solid bodies was thoroughly studied (see, e.g., Sternin and Shraiber, 1994) and can be taken into account, when necessary; therefore, below, we will not consider the particle disintegration in order not to overload the problem of modeling the disk evolution with details. In this case, the number density $n_{d,k}$ of fraction $k$ can change only through a decrease in the number of particles of this fraction as they combine with other dust particles and through an increase in the number of particles of this fraction due to the adhesion of smaller condensates. The system of kinetic equations that describes the coagulation can then be written as (Smoluchowski, 1936)

$$\overset{\circ}{n}_{d,k} = \frac{1}{2}\sum_{j=1}^{k-1} K_{j(k-j)} n_{d,j} n_{d,(k-j)} - \sum_{j=1}^{m} K_{kj} n_{d,k} n_{d,j} \quad (k = 1, 2, \ldots, m), \tag{27}$$

where $\overset{\circ}{n}_{d,k}(\mathbf{x}, t)$ is the total rate of change in the number density $n_{d,k}(\mathbf{x}, t)$ of fraction-$k$ dust particles via the coagulation processes; $K_{kj}(d_k, d_j)$ is the coagulation coefficient (kernel) for particles of sizes $k$ and $j$ that characterizes the coagulation interaction efficiency; it is defined as the mean number of collisions between particles of sizes $d_k$ and $d_j$ in a unit volume per unit time for a unit number density of one and the other type. Since such an interaction between two different-size particles in the flow is complicated by the influence of the ambient medium, the pattern of interaction in a laminar or turbulent flow, and the force fields (gravitation, electromagnetic field, molecular interaction), determining the coagulation kernel is a challenging problem of its own (see, e.g., Voloshchuk, 1984; Mazin, 1971). Kolesnichenko (2001) analyzed various coagulation mechanisms[20] for a turbulized gas–dust cloud and provided the corresponding expressions for the coefficients $K_{kj}$. Given (27), the system of instantaneous equations for the conservation of the number of fraction-$k$ dust particles takes the form

$$\frac{\partial}{\partial t} n_{d,k} + \nabla \cdot (n_{d,k}\mathbf{u}_d) = \overset{\circ}{n}_{d,k} = \frac{1}{2}\sum_{j=1}^{k-1} K_{j(k-j)} n_{d,j} n_{d,(k-j)} - n_{d,k}\sum_{j=1}^{m} K_{kj} n_{d,j} \quad (k = 1, 2, \ldots, m). \tag{28}$$

The balance equation for the total number $N'_d = \sum_k n_{d,k}$ of disperse particles in a unit total gas-suspension volume determined only by the coagulation processes (see (15)) follows from (28):

$$\rho\frac{d}{dt}\left(\frac{N'_d}{\rho}\right) + \nabla \cdot (N'_d \mathbf{w}_d) = \sum_k \overset{\circ}{n}_{d,k} = -\frac{1}{2}\sum_{k=1}\sum_{j=1} K_{kj} n_{d,k} n_{d,j}; \tag{29}$$

the right-hand side of Eq. (29) is equal to half the second term on the right-hand side of Eq. (28), since the total number of dust particles in a unit volume does not increase during the coagulation. In the spatially uniform case where all coagulation constants are approximately equal, $K_{kj} = K$, Eq. (29) $\partial N'_d \partial t = -(K/2)N'^2_d$ (with the initial condition $N'_d(0) = N'_{d0}$ has a simple solution, $N'_d(t) = N'_{d0}/(1 + qt)$, where $q = KN'_{d0}/2$, which allows the coagulation constant $K$ to be determined experimentally (from the slope of the straight line).

Given that the total mass of the dust particles during the coagulation is conserved, $\sum_k \mathcal{M}_{d,k}\overset{\circ}{n}_{d,k} = 0$, the summation of the left- and right-hand sides of the equations of system (28) over $k$ that were first multiplied by the mass of an individual fraction-$k$ particle, $\mathcal{M}_{d,k} = \rho_d U_1 k$, leads to Eq. (14*), which allows the total dust volume concentration $s$ in a two-phase polydisperse flow to be calculated.

It is also important to keep in mind that the number of nonlinear differential equations (28) required to describe the space-time distribution of the entire set of dust particle sizes in the disk is generally infinite. At the same time, we have to use a finite ($m$) number of equations when numerically simulating the coagulation processes based on system (28). Of course, the "loss of material" is possible in this case, since a number of particles can coagulate to sizes exceeding the largest size $d_{d,m}$ taken into account in this approach. Therefore, for our purposes, a different, integral form of the system of coagulation equations (28) is preferred.

To obtain this form, we assume that the number of particles with the volume from $U$ to $U + dU$ located at time $t$ in a volume element in the vicinity of point $\mathbf{x}$ is

[20]The coagulation of particles in a gas–dust flow can be caused by the simultaneous action of various particle collision mechanisms. These primarily include the gravitational coagulation, electric coagulation, Brownian coagulation, turbulent coagulation, and their various combinations like the turbulent–Brownian coagulation of charged and neutral particles, the Brownian coagulation of charged particles in a gravitational field, etc.

$f(U, \mathbf{x}, t)\mathrm{d}U$. By definition, the function $f(U, \mathbf{x}, t)$, which characterizes the particle size spectrum, satisfies the normalization relation

$$N_\mathrm{d}(\mathbf{x}, t) = \int_0^\infty f(U, \mathbf{x}, t)\mathrm{d}U. \quad (30)$$

Clearly, the formula

$$s(\mathbf{x}, t) = \int_0^\infty U f(\mathbf{x}, t, U)\mathrm{d}U \quad (31)$$

defines the total volume concentration of the dust particles. Since the volume of the size-$k$ particles is equal to $kU_1$, the number density $n_{\mathrm{d},k}$ of particles $k$ can be expressed in terms of $f(\mathbf{x}, t, U)$ as

$$n_{\mathrm{d},k} = f(kU_1, \mathbf{x}, t)U_1. \quad (32)$$

Using this relation, we can derive the following kinetic coagulation equation from (28) after the operation $U_1 \longrightarrow \mathrm{d}U$:

$$\frac{\partial f(U, \mathbf{x}, t)}{\partial t} + \nabla \cdot [f(U, \mathbf{x}, t)\mathbf{u}_\mathrm{d}]$$
$$\equiv \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{f(U, \mathbf{x}, t)}{\rho}\right) + \nabla \cdot [f(U, \mathbf{x}, t)\mathbf{w}_\mathrm{d}]$$
$$= \frac{1}{2}\int_0^U f(W, \mathbf{x}, t)f(U - W, \mathbf{x}, t)K(W, U - W)\mathrm{d}W \quad (33)$$
$$-f(U, \mathbf{x}, t)\int_0^\infty f(W, \mathbf{x}, t)K(W, U)\mathrm{d}W,$$

which is a generalization of the well-known Muller equation for describing a coagulating disperse medium (see, e.g., Voloshchuk, 1984) to the spatially nonuniform motions of a gas suspension. Here, $K(W, U)$ is the symmetric (in arguments) coagulation kernel that determines the behavior of a dispersed medium in time. To solve this equation, we must require the satisfaction of the conditions $f(U, \mathbf{x}, t) \longrightarrow 0$ for $U \longrightarrow 0$ and $U \longrightarrow \infty$ and specify the initial condition $f(U, \mathbf{x}, 0) = f_0(U, \mathbf{x})$ and the boundary conditions.

The kinetic equation (33) is a nonlinear integro-differential equation whose solution can generally be obtained only by numerical methods, since, unfortunately, the terms that describe the convection of dust particles severely complicate the standard coagulation equation (see, e.g., Lissauer and Stewart, 1993). Several exact analytical solutions of the nonstationary spatially uniform analog of Eq. (33) for some of the structurally simple coagulation kernels (linear in each individual argument) based on the Laplace integral transform are known in the literature (see, e.g., Safronov, 1969; Voloshchuk, 1984). Therefore, the following should be noted. The analyses of the coagulation processes for kernels $K(W, U) = \Lambda_0$ that do not depend on the coagulating-particle volumes are currently most advanced theoretically. The solution of the coagulation equation with the kernel $K(W, U) = \Lambda_1 WU$ can hardly be considered physically feasible, since it is not continuous in time (starting from a certain time, the number of particles in the system becomes negative (Voloshchuk, 1984). An analytical solution of the kinetic equation with a kernel proportional to the sum of the coagulating-particle volumes, $K(W, U) = \Lambda_2(W + U)$, was obtained by Safronov (1969) when studying the evolution of a protoplanetary gas–dust cloud. However, as yet no disperse system for which the coagulation microphysics would exactly lead to kernels of such a type has been found.

At the same time, when hydrodynamically modeling a gas–dust disk, the full knowledge of the particle size distribution function is often not required, and only information about the behavior of its first several moments of type $N_\mathrm{d}(\mathbf{x}, t)$, $s(\mathbf{x}, t)$, and the like in time and space will suffice. In this case, one of the possible approximate methods for solving the kinetic coagulation equation, in particular, the method of moments, can be used. In Appendix A, we illustrate the potentialities of this method by solving the kinetic coagulation equation (33) for the case where the particle size distribution depends on one space coordinate $z$, which corresponds to steady dust motion when solid particles settle under gravity to the subdisk.

### *The Conservation of Total Momentum*

In modeling a protoplanetary cloud, we have to solve the equations of radiation hydrodynamics for large space-time scales of motion that define the averaged thermohydrodynamic and radiation parameters of the gas–dust disk medium. When the linear size of the total volume element $\delta\mathcal{V}$ is much larger than the radiation mean free path $\lambda_\mathrm{rad}$, the radiation energy and pressure cannot be disregarded. It is quite clear that in the case of local equilibrium between radiation and matter, where the radiation energy density is $E_\mathrm{rad} = aT^4/\rho$ (per unit mass) and the radiation pressure is

$$p_\mathrm{rad} = \frac{\rho E_\mathrm{rad}}{3} = \frac{1}{3}aT^4, \quad (34)$$

we should everywhere add the radiation energy and pressure to the internal energy $E(\mathbf{x}, t)$ and the thermal pressure $p(\mathbf{x}, t)$ in the equations of heterogeneous mechanics and consider the process of radiative heat conduction. Here, $a = 4\sigma/c$, $\sigma$, and $c$ are the radiation density constant, the Stefan–Boltzmann constant, and the speed of light, respectively.

The instantaneous equation for the conservation of total momentum of the gas–dust material can be derived, for example, by adding the equations of motion for the individual phases (17*). As a result, the differential equation for the momentum conservation of the disk medium as a whole (including the radiation field), which, in contrast to the continuity equation (13), depends on the relative motion of the phases, can be written as

$$\rho\frac{d\mathbf{u}}{dt} \equiv \frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\mathbf{u}) = -\nabla p_{\text{sum}} + \nabla\cdot\mathbf{\Pi}^*_{\text{sum}} + \rho\mathbf{g}, \qquad (35)$$

where $\nabla\cdot(\mathbf{ab}) \equiv \sum_k\sum_l \mathbf{i}_l\frac{\partial(a_k b_1)}{\partial x_k}$ is the divergence of the dyad $\mathbf{ab} = \sum_k\sum_l \mathbf{i}_l\mathbf{i}_k(a_k b_1)$ (see Appendix B); $p_{\text{sum}}(\mathbf{x}, t)$ is the total pressure equal to the sum of the thermal pressure of the gas–dust mixture and the radiation pressure, $p_{\text{sum}} = p + p_{\text{rad}}$;

$$\mathbf{\Pi}^*_{\text{sum}} \equiv \mathbf{\Pi}_{\text{sum}} + \mathbf{\Pi}_{\text{rel}} = \mathbf{\Pi}_{\text{g}} + \mathbf{\Pi}_{\text{rad}} - (1-s)\rho_{\text{g}}\mathbf{w}_{\text{g}}\mathbf{w}_{\text{g}} - s\rho_{\text{d}}\mathbf{w}_{\text{d}}\mathbf{w}_{\text{d}}; \qquad (36)$$

$\mathbf{\Pi}_{\text{sum}}(\mathbf{x}, t)$ is the total viscous stress tensor[21] equal to the sum of the viscous stress tensors for a heterogeneous mixture, $\mathbf{\Pi} = \sum_\alpha \mathbf{\Pi}_\alpha \cong \mathbf{\Pi}_{\text{g}}$ (since we assumed that $\mathbf{\Pi}_{\text{d}} \cong 0$), and the radiative shear stress tensor $\mathbf{\Pi}_{\text{rad}}$; $\mathbf{\Pi}_\alpha$ is the viscous stress tensor for phase $\alpha$, which depends on the deformation rate tensor determined by the velocity field of the corresponding phase;

$$\mathbf{\Pi}_{\text{rel}} \equiv -\sum_\alpha \tilde{\rho}_\alpha\mathbf{w}_\alpha\mathbf{w}_\alpha = -(1-s)\rho_{\text{g}}\mathbf{w}_{\text{g}}\mathbf{w}_{\text{g}} - s\rho_{\text{d}}\mathbf{w}_{\text{d}}\mathbf{w}_{\text{d}} \qquad (37)$$

is the "relative" stress tensor[22] that arises from the dynamical effects of the relative motion of solid particles and gas (see, e.g., Kolesnichenko and Maksimov, 2001); $\mathbf{g}(\mathbf{x}, t) = -\nabla\Psi$ is the vector of acceleration by the external bulk force (gravity); $\Psi(\mathbf{x}, t)$ is the Newtonian gravitational potential. When the mass of the gas–dust cloud accounts for a few percent of the mass of the central body or, more precisely, when $\mathcal{M}_{\text{disk}}/\mathcal{M}_\odot \le h_{\text{disk}}/R$, where $h_{\text{disk}}$ and $R$ are the disk half-thickness and radius, respectively (see, e.g., Hersant *et al.*, 2004), the dust particle self-gravity may be ignored; in this case, we have

$$\Psi = \frac{G\mathcal{M}_\odot}{|\mathbf{r}|}, \quad \mathbf{g} = -\nabla\Psi = \frac{G\mathcal{M}_\odot}{|\mathbf{r}|^3}\mathbf{r}, \qquad (38)$$

where $\mathcal{M}_\odot$ is the mass of the central body (star); $G$ is the gravitational constant; $|\mathbf{r}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the central radius vector, $\mathbf{r} = \sum_k \mathbf{i}_k x_k$; in what follows, the center of mass of the protostar is taken as the coordinate origin. In the cases where the self-gravity effects are important,

$$\Psi = G\mathcal{M}_\odot/|\mathbf{r}| + \Psi_{\text{cr}} \qquad (38^*)$$

and the self-gravity potential $\Psi_{\text{cr}}$ satisfies the Poisson equation $\nabla^2\Psi_{\text{cr}} = 4\pi G\rho$, where $\nabla^2 = \nabla\cdot\nabla$ is the Laplace operator.

The relative stress tensor $\mathbf{\Pi}_{\text{rel}}$ for a gas–dust disk can be written in several equivalent forms convenient for writing the model equations of motion in various coordinate systems. Using (6) and (12), we have

$$\begin{aligned}\mathbf{\Pi}_{\text{rel}} &\equiv -(1-s)\rho_{\text{g}}\mathbf{w}_{\text{g}}\mathbf{w}_{\text{g}} - s\rho_{\text{d}}\mathbf{w}_{\text{d}}\mathbf{w}_{\text{d}} \\ &= -(1-s)\rho_{\text{g}}\mathbf{u}_{\text{g}}\mathbf{u}_{\text{g}} - s\rho_{\text{d}}\mathbf{u}_{\text{d}}\mathbf{u}_{\text{d}} + \rho\mathbf{u}\mathbf{u} \\ &= -s\rho_{\text{d}}C_{\text{g}}\mathbf{w}\mathbf{w} \\ &= -s\rho_{\text{d}}C_{\text{g}}(\mathbf{i}_1\mathbf{i}_1 w_1 w_1 + \mathbf{i}_1\mathbf{i}_2 w_1 w_2 + \mathbf{i}_1\mathbf{i}_3 w_1 w_3 \\ &\quad + \mathbf{i}_2\mathbf{i}_1 w_2 w_1 + \mathbf{i}_2\mathbf{i}_2 w_2 w_2 + \mathbf{i}_2\mathbf{i}_3 w_2 w_3 \\ &\quad + \mathbf{i}_3\mathbf{i}_1 w_3 w_1 + \mathbf{i}_3\mathbf{i}_2 w_3 w_2 + \mathbf{i}_3\mathbf{i}_3 w_3 w_3).\end{aligned} \qquad (39)$$

It is important to note that when the dynamics of a protoplanetary cloud is modeled, these additional stresses must be taken into account when fractions of relatively large solid particles (≥1 mm) are present in it, since in this case there is a significant velocity difference between the phases, i.e., the relative velocity of the phases $\mathbf{w}$ can be equal in order of magnitude to the hydrodynamic velocity of the total continuum $\mathbf{u}$. At the same time, for very small particles ($\ll$1 mm at a Stokes number Stk $\ll 1$; see Eq. (100)), when the particles have time to react to a change in the parameters of the carrier medium, the approximation of a passive admixture can be used[23]—a two-phase gas–dust flow is approximated by a flow of a single-phase (generally multicomponent)

[21] The viscous stress tensor is a tensor of the second rank or a dyad (see Appendix B).

[22] In heterogeneous media, the laws that describe the relative motion of the phases become more complicated, because this motion is determined not by the diffusion mechanism (collisions between molecules during their random motion), but by the interaction between the phases as macroscopic systems. These processes can be described using forces and a more consistent allowance for the phase inertia. The relative stress tensor in the total equation of motion for the mixture leads to a cardinal difference between heterogeneous mechanics and multicomponent mechanics, for which the terms containing quantities of the second order relative to the diffusion velocities $\mathbf{w}_\alpha$ may be disregarded (the so-called diffusion approximation in the mechanics of mixtures.

[23] In the other extreme case (at Stk $\gg$ 1), where the large solid particles in the disk system do not change their state as the gas parameters vary, we can also consider a single-phase flow, but already of pure gas; the inverse effect of the large bodies can be taken into account by introducing distributed sources of resistance. Finally, when $C_{\text{d}} \ll 1$, the presence of rare particles of the gas–dust mixture does not affect the gas flow parameters and therefore, the approximation of a single particle can be used; here, first, the equations of motion for the gas are solved, and then the particle trajectories and the change in their state along the trajectories are determined from known gas parameters (see, e.g., Garaud *et al.*, 2005).

medium with certain effective thermophysical properties (density, gas constant, specific heat, etc.) (see Kolesnichenko, 2000). Below, we will use mainly the representation of the tensor $\mathbf{\Pi}_{rel}$ via the velocity vector of the relative phase motion, $\mathbf{w} \equiv \mathbf{u}_d - \mathbf{u}_g$.

Since the radiative shear stress tensor $\mathbf{\Pi}_{rad}$ is known (see, e.g., Tassoul, 1979) to be structurally similar to the viscous stress tensor for material $\mathbf{\Pi}$, we may write[24]

$$\mathbf{\Pi}_{sum} = (\mathbf{\Pi} + \mathbf{\Pi}_{rad}) \cong 2(\mu_g + \mu_{rad})\overset{\circ}{\mathbf{D}} + (\xi_g + 5/3\mu_{rad})(\nabla \cdot \mathbf{u})\mathbf{I}, \quad \overset{\circ}{\mathbf{D}} = \mathbf{D} - 1/3\mathbf{I}\nabla \cdot \mathbf{u}, \tag{40}$$

where $\overset{\circ}{\mathbf{D}}$ is the deformation rate tensor; $\mathbf{D} = 1/2(\nabla\mathbf{u} + (\nabla\mathbf{u})^{transp})$ is the deformation tensor; $\mathbf{I}$ is a unit vector (or a unit dyad, $\mathbf{I} = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3$); $\mu_g(\rho, T)$ and $\xi_g(\rho, T)$ are the molecular coefficients of dynamic and bulk viscosities of the gas, respectively; $\mu_{rad} = 4aT^4/15c\tilde{\kappa}\rho$ is the coefficient of radiative viscosity; $\tilde{\kappa}$ is the total Rosseland mean opacity of the medium, which, in turn, also depends on $\rho$, $s$, $N_d$, $T$, and the chemical composition of the gas (see Eqs. (72) and (73)).

### *The Conservation of Internal Energy*

The instantaneous heat influx equation (the equation of internal energy) for a heterogeneous gas–dust medium as a whole under the above assumptions can be written as (Kolesnichenko and Maksimov, 2001)

$$\rho\frac{d}{dt}(E_{sum}) = -\nabla \cdot \mathbf{q}_{sum} - p_{sum}\nabla \cdot \mathbf{u} + \Phi_u + \sum_\alpha \mathbf{J}_\alpha \cdot \mathbf{K}_\alpha. \tag{41}$$

Here, $\mathbf{K}_\alpha \cong -\frac{d_\alpha \mathbf{u}_\alpha}{dt} + \mathbf{F}_\alpha - \frac{1}{2\tilde{\rho}_\alpha}\sigma_{\alpha\beta}\mathbf{w}_{\alpha\beta}$ is the generalized thermodynamic diffusion force that includes the "inertial term" and the term due to phase transitions (see (18)); $E_{sum} = E + E_{rad}$ is the total internal energy of the disk system (matter plus radiation) per unit mass; $E(\mathbf{x}, t) \equiv \sum_\alpha C_\alpha e_\alpha$ is the internal energy of the material;[25] $e_\alpha(\mathbf{x}, t)$, $h_\alpha(\mathbf{x}, t)$ $(=e_\alpha + p/\rho_\alpha)$ are, respectively, the partial internal energy and enthalpy (per unit mass) of the phase-$\alpha$ material; $E_{rad}$ is the radiation energy density (per unit mass) defined by the Stefan–Boltzmann law, $E_{rad} = aT^4/\rho$; $\mathbf{q}_{sum} = \mathbf{q} + \mathbf{q}_{rad}$ is the total energy flux density in the system; $\mathbf{q}_{rad}$ is the specific radiatively transferred energy flux;[26] $\mathbf{q}$ is the specific energy flux related to the thermal motion of the particles of the phase material (i.e., determined by the heat conductivity) and to the transfer of particle enthalpies by phase diffusion flows: $\Phi_u \cong \mathbf{\Pi}_{sum} : \nabla\mathbf{u}$ is the dissipative function, the rate at which heat is generated by the viscous friction of the gas in a unit volume per unit time. In writing (41), we assumed that the thermodynamic functions (internal energy, enthalpy, etc.) are additive in the masses of the phases in the heterogeneous system, which is admissible only when the contribution from the near-surface (Knudsen) layer of solid particles is disregarded.

Using (17) and (18), the last term in Eq. (41) can be rewritten as $\sum_\alpha \mathbf{J}_\alpha \cdot \mathbf{K}_\alpha = \sum_\alpha \mathbf{w}_\alpha \cdot (-\mathbf{d}_\alpha + s_\alpha\nabla p)$, where $\mathbf{d}_g = -\mathbf{d}_d = R_{gd}\mathbf{w}$ (without thermophoresis). For the additional source of heat associated with the dissipation of kinetic diffusion energy, we will then have (an analog of the Joule heating for plasma)

$$\sum_\alpha \mathbf{J}_\alpha \cdot \mathbf{K}_\alpha = -C_d\mathbf{w} \cdot (-\mathbf{d}_g + s_g\nabla p) + C_g\mathbf{w} \cdot (-\mathbf{d}_d + s_d\nabla p) = R_{gd}|\mathbf{w}|^2 = s\sigma\mathbf{w} \cdot \nabla p,$$

since $s^2 \ll 1$. Here, $\sigma = (\rho_d - \rho_g)/\rho$ is the relative excess of the dust particle density above the gas density; for small solid particles, $s\sigma \ll 1$ and the last term in this relation may be neglected.

It is important to note that the heat influx equation (41) contains the true internal energy of the gas–dust medium $E$ per unit mass, which was determined by subtracting the potential and kinetic energies of all phases from the total energy $U_{tot}$ of the material of the disk system (Kolesnichenko and Maksimov, 2001),

$$E = U_{tot} - \sum_\alpha C_\alpha\Psi_\alpha - \sum_\alpha \frac{1}{2}C_\alpha|\mathbf{u}_\alpha|^2 = U_{tot} - \Psi - \frac{1}{2}|\mathbf{u}|^2 - \sum_\alpha C_\alpha\frac{1}{2}|\mathbf{w}_\alpha|^2 = U_{tot} - \Psi - \frac{1}{2}|\mathbf{u}|^2 - C_dC_g|\mathbf{w}|^2/2. \tag{42}$$

At the same time, if the internal energy of the gas–dust system is defined by the relation $E^* = U_{tot} - \Psi - 1/2|\mathbf{u}|^2$, then it will also include the macroscopic kinetic energy of the phases in the center-of-mass system, i.e., $E^* = E + C_dC_g|\mathbf{w}|^2/2$. If we now write Eq. (41) via the inter-

[24]If the matter–radiation interaction up to the terms of the lowest order in $|\mathbf{u}|/c$ is taken into account, then the following terms also enter into the radiative shear stress tensor components $(\mathbf{\Pi}_{rad})_{ik}$: $-c^{-2}(u_i(\mathbf{q}_{rad})_k + u_k(\mathbf{q}_{rad})_i + \delta_{ik}u_s(\mathbf{q}_{rad})_s)$, where $\mathbf{q}_{rad}$ is the radiative heat flux vector defined by Eq. (48) (see, e.g., Hazlehurst and Sargent, 1959).

[25]The internal energy of the gas–dust mixture as a whole that we introduced is its true internal energy, since it does not contain the contribution from the kinetic energy of the interphase diffusion.

[26]The radiative energy transfer should always be taken into account, since it is large even at a low radiation energy density (due to the high photon velocity).

nal energy $E^*$ redefined in this way, then it will take a usual form, i.e., it will not contain the terms $R_{gd}|\mathbf{w}|^2 = s\sigma\mathbf{w}\cdot\nabla p$. Indeed, using (19*) and the vector transformation $(\mathbf{a}\cdot(\mathbf{b}\cdot\nabla)\mathbf{c} = \mathbf{ab} : \nabla\mathbf{c})$, we can write the balance equation for the kinetic energy of the interphase diffusion as

$$\rho\frac{d}{dt}(C_d C_g|\mathbf{w}|^2/2) \approx \rho C_d C_g \frac{d}{dt}(|\mathbf{w}|^2/2) \cong -R_{gd}|\mathbf{w}|^2 + \mathbf{\Pi}_{rel} : \nabla\mathbf{u} + s\sigma\mathbf{w}\cdot\nabla p, \tag{43}$$

since, to the terms of the second order relative $\mathbf{w}$, we have

$$\rho\frac{d}{dt}(C_d C_g|\mathbf{w}|^2/2) = \rho C_d C_g \frac{d}{dt}(|\mathbf{w}|^2/2) + (|\mathbf{w}|^2/2)\left\{C_d\rho\frac{dC_g}{dt} + C_g\rho\frac{dC_d}{dt}\right\}$$
$$= \rho C_d C_g \frac{d}{dt}(|\mathbf{w}|^2/2) + (|\mathbf{w}|^2/2)(C_d - C_g)\nabla\cdot(\rho C_g C_d\mathbf{w}) \approx \rho C_d C_g\frac{d}{dt}(|\mathbf{w}|^2/2).$$

Still, the quantity $E$ in the heat influx equation (41) probably more deserves the name "internal energy" than $E^*$, since the internal energy must contain only the contribution from the thermal motion and the short-range molecular interactions and no macroscopic terms (see de Groot and Mazur, 1964).

**Other forms of the energy equation for a gas suspension.** Below, we will need the energy equations written in several other forms. Let us introduce the total enthalpy $H_{sum} = H + H_{rad}$ of the matter and radiation in the disk, where

$$\begin{cases} H \equiv \sum_\alpha C_\alpha h_\alpha = \sum_\alpha C_\alpha(e_\alpha + p/\rho_\alpha) = E + p/\rho \\ H_{rad} = E_{rad} + p_{rad}/\rho = 4/3aT^4/\rho \\ H_{sum} = E_{sum} + p_{sum}/\rho. \end{cases} \tag{44}$$

Using Eq. (42) and the transformation $\rho dE_{sum}/dt + p_{sum}\nabla\cdot\mathbf{u} = \rho dH_{sum}/dt - dp_{sum}/dt$, which is a corollary of definitions (44) and the mixture continuity equation (13), we will then obtain

$$\rho\frac{dH_{sum}}{dt} = \frac{dp_{sum}}{dt} - \nabla\cdot\mathbf{q}_{sum} + \Phi_u + R_{gd}|\mathbf{w}|^2 - s\sigma\mathbf{w}\cdot\nabla p. \tag{45}$$

This equation corresponds to the first law of thermodynamics (i.e., the law of conservation of thermal energy).

Let us now rewrite Eq. (45) in the variables $T(\mathbf{x}, t)$ and $p(\mathbf{x}, t)$. For most of the purposes pertaining to our problem of modeling the evolution of an accretion disk, it will suffice to approximate the partial enthalpies of the gas and dust (per unit mass) using the expressions $h_g = c_{Pg}T + h_g^0$, and $h_d = c_{Pd}T + h_g^0$, where $h_\alpha^0$ is the enthalpy of phase $\alpha$ at zero temperature (the so-called heat of formation) and $c_{P\alpha}$ is the specific heat (at constant pressure) of phase $\alpha$. Below, the thermophysical quantities $c_{P\alpha}$ and $h_\alpha^0$ are assumed to be the constants that approximate the actual disk specific heats $c_{P\alpha}(T)$ and partial heats of formation $h_\alpha^0(T)$ in a limited temperature range. We can then write

$$H = c_P T + \sum_\alpha C_\alpha h_\alpha^0, \qquad H_{rad} = E_{rad} + p_{rad}/\rho = 4/3aT^4/\rho, \tag{46}$$

where $c_P = \sum_\alpha c_{P\alpha}C_\alpha = \rho^{-1}\{\rho_g(1-s)c_{Pg} + s\rho_d c_d\}$ is the total specific heat of the "gas–solid particles" system at constant pressure. Using now Eqs. (46), along with Eqs. (9), (12), and (13), we obtain

$$\rho\frac{dH}{dt} \equiv \rho\sum_\alpha C_\alpha\frac{dh_\alpha}{dt} + \rho\sum_\alpha h_\alpha\frac{dC_\alpha}{dt} = \rho c_P\frac{dT}{dt} + \sum_\alpha h_\alpha(-\nabla\cdot J_\alpha + \sigma_{\alpha\beta})$$
$$= \rho c_P\frac{dT}{dt} - \nabla\cdot\left[\sum_\alpha h_\alpha\mathbf{J}_\alpha\right] + \sum_{\rho=1}^{r} q_\rho\xi_\rho + \nabla T\cdot\left[\sum_\alpha c_{P\alpha}\mathbf{J}_\alpha\right], \tag{47}$$

where the relation $q_\rho \equiv \sum_\alpha h_\alpha\nu_{\alpha,\rho} = q_\rho^0 + \sum_\alpha c_{P\alpha}\nu_{\alpha,\rho}$ introduces the so-called heat of reaction $\rho$, which is equal to the difference between the products of the partial enthalpies of the reaction products by the corresponding stoichiometric coefficients and the analogous sum for the reactants ($\nu_{\alpha,\rho} \equiv \sum_{k=1}^N M_{(k)}\nu_{\alpha(k),\rho}$); note that $q_\rho^0 = \sum_\alpha h_\alpha^0\nu_{\alpha,\rho}$ can be interpreted as the heat of phase transition $\rho$ at zero temperature. The last term on the right-hand side of Eq. (47) represents the effect of the so-called "diffusing specific heats," which is negligible and, hence, is commonly ignored.

According to Kolesnichenko and Maksimov (2001), the total energy flux $\mathbf{J}_q \equiv \mathbf{q} - \sum_\alpha h_\alpha\mathbf{J}_\alpha$ related to the thermal motion of particles in a heterogeneous continuum[27] can be written in the standard form[28]

$$\mathbf{J}_q \equiv \mathbf{q} - \sum_\alpha h_\alpha \mathbf{J}_\alpha \equiv \mathbf{q} - \rho C_g C_d (h_d - h_g)\mathbf{w} = -\chi_g \nabla T,$$

or

$$\mathbf{q} = \sum_\alpha h_\alpha \mathbf{J}_\alpha - \chi_g \nabla T, \qquad (48)$$

which generalizes the analogous relation for multicomponent homogeneous mixtures (see Hirschfelder *et al.*, 1954) to heterogeneous media. In a similar way, by excluding the disk regions close to the surface of the protostar, we can write the law of heat conduction for the radiative heat flux vector,

$$\mathbf{q}_{\rm rad} = -\chi_{\rm rad} \nabla T. \qquad (48^*)$$

Here, $\chi_g$ is the molecular coefficient of heat conductivity for the gas, and $\chi_{\rm rad} = 4acT^3/(3\tilde{\kappa}\rho)$ is the coefficient of radiative (nonlinear) heat conductivity, which strongly depends on the temperature and density of the material (see Eq. (71)).

Substituting (47), (48), and (48*) in Eq. (44) ultimately yields

$$\rho c_{\rm P,\,sum} \frac{dT}{dt} = \frac{dp_g}{dt} + \nabla \cdot (\chi_{\rm sum} \nabla T) - 4p_{\rm rad} \nabla \cdot \mathbf{u}$$
$$+ \Phi_u + R_{gd}\mathbf{w}^2 - s\sigma \mathbf{w} \cdot \nabla p_g - \sum_{\rho=1}^{r} q_\rho \xi_\rho, \qquad (49)$$

where we use the notation $\chi_{\rm sum} = \chi_g + \chi_{\rm rad}$ and $c_{\rm P,\,sum} = c_{\rm P} + 16aT^3/3\rho$.

Finally, let us derive the balance equation for the specific entropy $S = \sum_\alpha C_\alpha S_\alpha$ of the total continuum modeling the gas–dust disk medium as a whole, which is commonly called the general heat transfer equation (here, $S_\alpha$ is the entropy per unit mass of phase $\alpha$). For this purpose, we will use the fundamental Gibbs relation (see, e.g., Prigogine and Defay, 1954) for a single-temperature heterogeneous multicomponent radiative continuum in the single-pressure approximation. Being written along the center-of-mass trajectory of the volume element $\delta\mathcal{V}$, it takes the form (Kolesnichenko and Maksimov, 2001)

$$T\frac{dS_{\rm sum}}{dt} = \frac{dE_{\rm sum}}{dt} + p_{\rm sum}\frac{d}{dt}\left(\frac{1}{\rho}\right)$$
$$- \sum_\alpha \sum_{k=1}^{n} \mu_{\alpha(k)} \frac{d}{dt}\left(\frac{s_\alpha n_{\alpha(k)}}{\rho}\right), \qquad (50)$$

where $\mu_{\alpha(k)}$ is the chemical potential of component $k$ in phase $\alpha$. Using Eqs. (9), (13), and (41), we can write the Gibbs relation (50) in the form of a balance equation,

$$\rho\frac{dS_{\rm sum}}{dt} + \nabla \cdot \left\{\frac{1}{T}\left(\mathbf{q}_{\rm sum} - \sum_\alpha G_\alpha \mathbf{J}_\alpha\right)\right\} = \sigma_{(S)}, \qquad (51)$$

where

$$0 \le T\sigma_{(S)} = -(\mathbf{J}_q + \mathbf{q}_{\rm rad})\frac{\nabla T}{T} - \sum_\alpha \mathbf{w}_\alpha \cdot \mathbf{d}_\alpha$$
$$+ \mathbf{\Pi}_{\rm sum} : \nabla \mathbf{u} + \sum_\alpha \mathbf{\Pi}_\alpha : \nabla \mathbf{w}_\alpha + \sum_{\rho=1}^{r} A_\rho \xi_\rho \qquad (52)$$

is the energy dissipation in irreversible processes, which is a local measure of nonequilibrium of the system; $G_\alpha = \rho_\alpha^{-1}\sum_{k=1}^{n} n_{\alpha(k)} = e_\alpha + p/\rho_\alpha - TS_\alpha$ is the Gibbs free energy of an elementary macroparticle of phase $\alpha$;

$$A_\rho \equiv -\sum_{\alpha=1} \sum_{k=1}^{n} \mu_{\alpha(k)} \nu_{\alpha(k),\,\rho} \qquad (53)$$

is the chemical affinity of reaction $\rho$ that generally proceeds between components in different phases.

Note that the specific representation of the rate of entropy production ($T\sigma_{(S)}$) as a bilinear form is used in nonequilibrium thermodynamics to establish the defining relations that linearly relate the thermodynamic fluxes and the conjugate thermodynamic forces in the irreversible process under consideration by the Onsager method. In particular, the generalized Stefan-Maxwell relations (17) for heterogeneous media were derived by Kolesnichenko and Maximov (2001) precisely in this way.

### *The Thermodynamic Equation of State*

Below, we will use a baroclinic equation of state for a mixture of perfect gases as the thermal state of the multicomponent gas phase of the disk (the equation for pressure):

$$p_g = \sum_{(k)} p_{(k)} = k_B T \sum_k n_{g(k)} = \rho_g \mathcal{R}_g T, \qquad (54)$$

[27]Recall that the heat flux for diffusing mixtures can be defined in various ways, with the specific form of the expression for the rate of entropy production $\sigma_{(S)}$ corresponding to each definition of the heat flux; the choice in each specific case depends on the convenience of considering the problem.

[28]In (48), we disregarded the thermophoretic effect, $k_{P\alpha} = 0$.

where $\mathcal{R}_g = k_B \sum_k n_{g(k)}/\rho_g = k_B/\mathcal{M}_g$; $\mathcal{M}_g$ is the mean molecular mass of the gas particles, which below is assumed to be constant.

Assuming that the partial pressures in the phases are equal, $p_g = p_d = p$, let us write the equation of state for the disk material as

$$p = \rho \mathcal{R} T, \quad \mathcal{R}(C_g, s) = \mathcal{R}_g \rho_g/\rho = \mathcal{R}_g C_g/(1-s) \cong \mathcal{R}_g C_g \tag{55}$$

(in the case considered here, $\mathcal{R}$ is not a constant). The approximate equality in Eq. (55) holds for a gas suspension with a low volume concentration of the condensed phase (i.e., when $s \ll 1$, which is assumed in this paper; nevertheless, the dynamical effect of solid particles on the gas flow may turn out to be significant even in this case due to the enormous influence of the gravitational force). Thus, the gas–dust disk medium as a whole may be treated as a perfect gas with the adiabatic index $\gamma$ and the speed of sound $c_s$ defined by the relations

$$\gamma \equiv \frac{c_P}{c_P - \mathcal{R}} \cong \frac{\rho_g c_{Pg} + s\rho_d c_{Pd}}{\rho_g(c_{Pg} - \mathcal{R}_g) + s\rho_d c_{Pd}}, \qquad 1 \le \gamma \le \gamma_g \equiv \frac{c_{Pg}}{c_{Pg} - \mathcal{R}_g}, \tag{56}$$

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_S = \gamma\frac{p}{\rho} = \gamma \mathcal{R} T \cong c_{sg}^2 \frac{\gamma \rho_g}{\gamma_g \rho}, \qquad c_s < c_{sg} \equiv \left(\frac{\partial p}{\partial \rho_g}\right)_{S_g} = (\gamma_g \mathcal{R}_g T)^{1/2}, \tag{57}$$

where $\gamma_g$ and $c_{sg}$ are the adiabatic index and the isothermal speed of sound in a pure gas. For a gas of solar composition, with hydrogen and helium comprising 98% of it, the adiabatic index is $\gamma_g = 1.45$ and the mean molecular mass is $\mathcal{M}_g = 2.39$.

### *Radiative Processes*

Radiative heat transfer has a decisive effect on the state and motion of a turbulized high-temperature protoplanetary cloud. Meanwhile, the interaction between radiative heat transfer and turbulence has been studied inadequately; as a result, it has been disregarded in the literature and in modeling the evolution of a gas–dust disk until recently. Since this interaction can actually be significant (see, e.g., Ievlev, 1975), below we attempt to take it into account (at least in part) using the approach being developed here. Therefore, let us consider in more detail some of the basic concepts of the theory of radiative transfer that we will need for the above purposes.

The emission and absorption of photons[29] are described by a radiative transfer equation that for a gas–dust medium in local equilibrium (at any point in space and at any instant in time) takes the form

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{\Omega} \cdot \nabla I_\nu = \rho \kappa_\nu (B_\nu - I_\nu). \tag{58}$$

Here, $I_\nu \equiv I_\nu(\mathbf{x}, \mathbf{\Omega}, t)$ is the spectral intensity of the radiation defined in such a way that $I_\nu d\nu d\mathbf{\Omega}$ describes the energy of the photons in the frequency range from $\nu$ to $\nu + d\nu$ that cross a unit surface element with normal $\mathbf{\Omega}$ within the solid angle $d\Omega$ oriented along $\mathbf{\Omega}$ per unit time; $B_\nu(T) \equiv (2h\nu^3/c^2)[\exp(h\nu/k_B T) - 1]^{-1}$ is the Planck function; $h$ is the Planck constant; $\kappa_\nu(\mathbf{x}, t)$ is the total spectral attenuation coefficient (opacity) expressed in terms of the cross-sections for the elementary physical processes in the gas–dust mixture as

$$\rho\kappa_\nu = (1-s)\sum_k n_{g(k)}\sigma_{(k)}(\nu) + N_d \frac{\pi \tilde{d}_d^2}{4} Q_d(m(\nu), \tilde{d}_d); \tag{59}$$

$\sigma_{(k)}(\nu) \equiv \sigma_{a(k)}(\nu)[1 - \exp(h\nu/k_B T)] + \sigma_{s(k)}^{\text{eff}}$ is the cross-section for the attenuation of radiation at frequency $\nu$ per one gas molecule of type $k$, which is equal to the photon absorption cross-section (corrected for the induced emission of radiation) plus the effective scattering cross-section; $Q_d = Q_{ds} + Q_{da}$; $Q_{ds}$ and $Q_{da}$ are, respectively, the efficiency factors for the scattering and absorption of light by dust particles (the dimensionless quantities calculated using the Mie theory); $m(\nu)$ is the complex refractive index of the grain material. The following moments used here are related to the function $I_\nu$:

$$E_{\text{rad},\nu}(\mathbf{x}, t) \equiv \frac{1}{c\rho}\int_{4\pi} I_\nu(\mathbf{x}, \mathbf{\Omega}, t) d\mathbf{\Omega}, \tag{60}$$

the spectral energy density (per unit mass); and

$$\mathbf{q}_{\text{rad},\nu}(\mathbf{x}, t) = \int_{4\pi} I_\nu(\mathbf{x}, \mathbf{\Omega}, t)\mathbf{\Omega} d\mathbf{\Omega}, \tag{61}$$

the spectral energy density along $\mathbf{\Omega}$. The total energy density and flux can be obtained by integrating the cor-

[29]The scattering is known to have no direct effect on the thermal regime of the medium. That is why, in general, the scattering of radiation is disregarded in the problems of radiation hydrodynamics (which we will do below), and only the true attenuation coefficient $\kappa_\nu$ and the true source function of the radiation $B_\nu$ (without scattering) are considered.

responding monochromatic quantities over the frequency,

$$E_{\text{rad}}(\mathbf{x}, t) \equiv \int_{\nu=0}^{\infty} E_{\text{rad},\nu}(\mathbf{x}, t)\mathrm{d}\nu,$$
$$\mathbf{q}_{\text{rad}}(\mathbf{x}, t) \equiv \int_{\nu=0}^{\infty} \mathbf{q}_{\text{rad},\nu}(\mathbf{x}, t)\mathrm{d}\nu. \tag{62}$$

Equation (58) written along the direction of propagation of the radiation takes a simpler form,

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}l} = \rho\kappa_\nu(B_\nu - I_\nu). \tag{63}$$

Here, $l$ is the coordinate along the ray; below, we will omit the term $c^{-1}\partial I_\nu/\partial t$ in Eq. (58), since the characteristic time scales for the motion of a gas–dust medium are much longer than $l^*/c$, where $l^*$ is the ray length. If the optical depth of the layer of the gas–dust medium (with the ray length $l$) along the direction of propagation of the radiation is defined by

$$\tau_\nu = \int_0^l \rho\kappa_\nu \mathrm{d}l, \tag{64}$$

then Eq. (63) can be easily integrated to give the following expression for the intensity of the radiation from a region with a total optical depth $\tau_\nu$:

$$I_\nu(\tau_\nu) = I_\nu(0)\exp(-\tau_\nu) + \int_{\tau_{\nu 1}=0}^{\tau_\nu} B_\nu(\tau_{\nu 1})\exp[-(\tau_\nu - \tau_{\nu 1})]\mathrm{d}\tau_{\nu 1}, \tag{65}$$

where $I_\nu(0)$ is the integration constant; the latter has the meaning of the radiation intensity at some point on the ray at which we set the coordinate $l$ equal to zero, $l = 0$. When removing the point $l = 0$ to a great distance, it follows from Eq. (65) that

$$I_\nu \approx \int_{-\infty}^{l} B_\nu(l_1)\exp\left(-\int_{l_2-l_1}^{l} \rho\kappa_\nu(l_2)\mathrm{d}l_2\right)\rho\kappa_\nu(l_1)\mathrm{d}l_1. \tag{66}$$

Ideally, Eqs. (65) and (66) allow the intensity $I_\nu(\mathbf{x}, \mathbf{\Omega}, t)$ at various points and in various directions to be found for known optical properties of the medium (i.e., the distribution $\kappa_\nu(\mathbf{x}, t)$ and boundary conditions in the midplane of the disk; the radiative heat flux distribution $\mathbf{q}_{\text{rad}}(\mathbf{x}, t)$ can then be calculated using Eqs. (61) and (62).

At the same time, the heat influx equations (41) and (45) include the divergence of the radiation flux $\nabla \cdot \mathbf{q}_{\text{rad}}$. For a known $I_\nu$ distribution, this quantity can often be found without calculating $\mathbf{q}_{\text{rad}}$ from (62). Integrating the steady-state equation (58) over the entire frequency spectrum and over the solid angle $\Omega$ yields the following general expression for the contribution of the radiation to the thermal balance of the gas–dust disk medium:

$$\nabla \cdot \mathbf{q}_{\text{rad}} = -\int_{\nu=0}^{\infty}\int_{4\pi} \mathbf{\Omega}\nabla I_\nu \mathrm{d}\Omega\mathrm{d}\nu = \int_{\nu=0}^{\infty}\left\{4\pi\rho\kappa_\nu B_\nu(T) - \int_{4\pi}\rho\kappa_\nu I_\nu \mathrm{d}\Omega\right\}\mathrm{d}\nu, \tag{67}$$

where the first and second terms correspond, respectively, to the spontaneously emitted and absorbed radiation energies in a unit volume per unit time. The contribution of the radiation to the heat influx equation (41) is generally difficult to calculate using formula (67). However, these calculations are simplified significantly in the following two cases that are important for modeling the various evolutionary stages of a protoplanetary gas–dust cloud.

(1) At small optical depths of the gas–dust disk. In this case, the term with $I_\nu$ in (67) may be ignored, i.e., we can assume in the heat influx equation that

$$\nabla \cdot \mathbf{q}_{\text{rad}} \cong 4\pi \int_{\nu=0}^{\infty} \rho\kappa_\nu B_\nu(T)\mathrm{d}\nu. \tag{68}$$

At high temperatures, this term can be significant even at a small optical depth of the gas (see Ievlev, 1975), for example, in the near-surface layer of the disk.

(2) At large optical depths of the gas–dust disk for the radiation of all energetically significant frequencies $\nu$. In this case, the diffusion approximation is applicable for the radiative heat transfer (the approximation of radiative heat conduction), where the radiation field $I_\nu$ is anisotropic only slightly. Multiplying the radiative transfer equation (58) by $\mathbf{\Omega}$ and integrating it by all angles yields (given that the isotropic term with $\rho\kappa_\nu B_\nu$ does not depend on the direction and, hence, does not contribute to the integral)

$$\int_{4\pi} \mathbf{\Omega}(\mathbf{\Omega}\cdot\nabla I_\nu)\mathrm{d}\Omega = -\rho\kappa_\nu \mathbf{q}_{\text{rad},\nu}; \tag{69}$$

whence we obtain for the total heat flux

$$\mathbf{q}_{\text{rad}} = -\int_{\nu=0}^{\infty} \frac{1}{\rho\kappa_\nu}\int_{4\pi} \mathbf{\Omega}(\mathbf{\Omega}\cdot\nabla I_\nu)\mathrm{d}\Omega\mathrm{d}\nu. \tag{70}$$

If we retain only the most significant isotropic part for the slightly anisotropic radiation field on the left-hand side of Eq. (69), then

$$\begin{aligned}\mathbf{q}_{\text{rad}} &= -\int_{\nu=0}^{\infty}\frac{1}{\rho\kappa_\nu}\int_{4\pi}\mathbf{\Omega}(\mathbf{\Omega}\cdot\nabla I_\nu)\mathrm{d}\Omega\mathrm{d}\nu \\ &\cong -\frac{c}{4\pi}\int_{\nu=0}^{\infty}\frac{1}{\rho\kappa_\nu}\left(\nabla B_\nu\int_{4\pi}\mathbf{\Omega}\right)\mathbf{\Omega}\mathrm{d}\Omega\mathrm{d}\nu \\ &= -\frac{c}{3\rho}\int_{\nu=0}^{\infty}\frac{1}{\kappa_\nu}\nabla B_\nu\mathrm{d}\nu = -\frac{c}{3\rho}\int_{\nu=0}^{\infty}\frac{1}{\kappa_\nu}\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\nabla T\mathrm{d}\nu \\ &= -\frac{4caT^3}{3\rho\tilde{\kappa}}\nabla T = -\chi_{\text{rad}}\nabla T,\end{aligned} \tag{71}$$

where we introduced the so-called total opacity of the medium $\tilde{\kappa}(\rho, s, T, N_d)$, which is defined as the Rosseland mean (for the inverse quantities $1/\kappa_\nu$) spectral opacity (see Pollack *et al.*, 1985),

$$\frac{1}{\tilde{\kappa}} = \frac{1}{4aT^3}\int_{\nu=0}^{\infty}(1/\kappa_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\mathrm{d}\nu = \frac{\int_{\nu=0}^{\infty}(1/\kappa_\nu)(\mathrm{d}B_\nu/\mathrm{d}T)\mathrm{d}\nu}{\int_{\nu=0}^{\infty}(\mathrm{d}B_\nu/\mathrm{d}T)\mathrm{d}\nu} \tag{72}$$

(since $\int_{\nu=0}^{\infty}(\mathrm{d}B_\nu/\mathrm{d}T)\mathrm{d}\nu = 4aT^3$). The diffusion approximation is valid if the radiation field is isotropic at distances comparable to or smaller than the photon mean free path, $\lambda_\nu = 1/\kappa_\nu$. Note also that Eq. (71) expresses the radiative flux vector $\mathbf{q}_{\text{rad}}$ in the inner regions of the gas–dust disk with an excellent accuracy. However, in the near-surface layers of the disk, the optical depth is of the order of or smaller than unity, and the flux is no longer defined by this local expression. Therefore, we must use the nonlocal solution (68) of the transfer equation, which is commonly used to study the stellar atmospheres.

**Optical properties of dust grains.** It is convenient to rewrite the spectral opacity of the medium related to the dust component that is defined by Eq. (59) as

$$\begin{aligned}\kappa_\nu(\rho, s, N_d) &= \frac{\pi\tilde{d}_d^2}{4\rho}N_d Q_d(m(\nu), d_d) \\ &= \frac{\pi^{1/3}6^{2/3}}{4\rho}s^{2/3}N_d^{1/3}[Q_{da}(m(\nu), d_d) + Q_{ds}(m(\nu), d_d)];\end{aligned} \tag{73}$$

this form explicitly depends on the first moments $s$ and $N_d$ (see (30) and (31)) of the dust particle size distribution function $f(U, \mathbf{x}, t)$. We are going to calculate the scattering and absorption of light by spherical solid bodies with a complex refractive index $m$ using the Mie theory. Note that, for example, $m = \infty$ corresponds to an infinite permittivity, $m = 1.33$ corresponds to ice particles (for visual wavelengths), $m = 1.33–0.09i$ corresponds to dirty ice (ice with absorbing admixtures), and $m = 1.27–1.37i$ corresponds to iron grains.

The size of a spherical solid particle is commonly expressed in terms of the dimensionless parameter $x(\nu) = \pi d_d/\lambda$, where $\lambda = c/\nu$ is the wavelength of light. For small $x$, the efficiency factor for the scattering of light by dust particles $Q_{ds}$ is very small; at $|mx| \ll 1$, we have the standard formula for Rayleigh scattering:

$$Q_{ds} = 8/3x^4|(m^2-1)/(m^2+2)|^2, \tag{74}$$

and the absorption efficiency factor in this case is given by

$$Q_{da} = -4x\mathrm{Im}[(m^2-1)/(m^2+2)], \tag{75}$$

where Im means that the imaginary part should be taken.

*The Basic System of Equations of Disk Hydrodynamics*

Let us summarize (for the convenience of referencing) the above equations of motion for a two-phase polydisperse gas–dust medium. These equations (the reference basis of the model), which include the relative motion of the phases, coagulation, phase transitions, and various physical–chemical and radiative processes, are intended, in particular, for a continuum description of the space-time evolution of the composition, dynamics, and thermal regime of a gas–dust cloud at the final laminar evolutionary stage of the protoplanetary disk (after the decay of turbulent motions[30]) in subdisk zones located at various distances from the proto-Sun (see, e.g., Nakagawa *et al.*, 1986). It is also important that these equations, which describe the instantaneous state of a turbulized protoplanetary cloud at any stage of its evolution, may be considered as the basic ones in studying the mean motion of the disk system when the probability-theoretical averaging of the stochastic equations of motion has to be performed to

[30] Turbulence may not decay completely in the disk regions close to the proto-Sun due to the disturbing effect of magnetic fields, corpuscular flows, etc. on the medium.

phenomenologically describe the hydrodynamic and physical–chemical processes. Thus, we have

$$\begin{cases} \dfrac{d\rho}{dt} + \rho\nabla\cdot\mathbf{u} = 0 \quad \left(\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) \\ \rho\dfrac{d}{dt}\left(\dfrac{s\rho_d}{\rho}\right) = -\nabla\cdot\mathbf{J}_d + \sum\limits_{\rho=1}^{r}\nu_{d,\rho}\xi_\rho \\ \left(\mathbf{J}_d = \rho C_d C_g \mathbf{w},\ C_g = 1 - \dfrac{s\rho_d}{\rho},\ \rho_d = \text{const}\right) \\ \rho\dfrac{d}{dt}\left(\dfrac{N_d}{\rho}\right) = -\nabla\cdot(N_d C_g \mathbf{w}) \\ -\dfrac{1}{2}\int\limits_0^\infty\int\limits_0^\infty K(W,U)f(W)f(U)dWdU + \sum\limits_k\sum\limits_{\rho=1}^{r}\nu_{d(k),\rho}\xi_\rho \\ \rho\dfrac{d\mathbf{u}}{dt} = -\nabla(p + p_{rad}) + \nabla(\mathbf{\Pi}_{sum} - \mathbf{J}_d\mathbf{w}) + \rho\dfrac{G\mathcal{M}_\odot}{|\mathbf{r}|^3}\mathbf{r} \\ (\rho c_{V_g} + 4aT^3/3)\dfrac{dT}{dt} \\ = -\nabla\cdot(\mathbf{J}_q + \mathbf{q}_{rad}) - (p + 4p_{rad})\nabla\cdot\mathbf{u} + \Phi_u \\ -s\rho_d C_g\dfrac{d}{dt}\left(\dfrac{\mathbf{w}^2}{2}\right) - \sum\limits_{\rho=1}^{r}q_\rho\xi_\rho \quad (\rho_g \cong \rho - s\rho_d) \\ p = p_g = \rho_g\mathcal{R}_g T, \quad p_{rad} = aT^4/3. \end{cases} \tag{76}$$

The hydrodynamic equations of motion (76) must be complemented by the corresponding expressions for the phase transition rates $\xi_\rho$ and the defining relations for the thermodynamic fluxes,

$$\begin{cases} \mathbf{\Pi}_{sum} = (\mu_g + \mu_{rad})[\nabla\mathbf{u} + (\nabla\mathbf{u})^{transp}] \\ + (\xi_g - 2/3\mu_g + \mu_{rad})(\nabla\cdot\mathbf{u})\mathbf{I} \\ \mathbf{w} \cong \dfrac{1}{\rho\theta_{dg}}\left(-\dfrac{d\mathbf{w}}{dt} - (\mathbf{w}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho_g}\nabla p_g\right) \\ \mathbf{J}_q = -\chi_g\nabla T, \quad \mathbf{q}_{rad} = -\chi_{rad}\nabla T, \end{cases} \tag{77}$$

as well as by the expressions for the coagulation coefficients $K(W, U)$ (see Kolesnichenko, 2001) and the coefficients of molecular transfer $\mu_g(s, T)$, $\xi_g(s, T)$, $\chi_g(s, T)$, $\theta_{dg}(s, N_d, \text{Re})$ and radiative heat conductivity $\chi_{rad}(s, N_d, T)$. For the above system of equations of two-phase mechanics (76)–(77), we must specify the initial and boundary conditions whose choice requires a special analysis in each specific case, since, in general, not the disk system as a whole with, say, such natural boundaries as the midplane of the disk or its outer boundary, but its separate regions are modeled.

It is important to emphasize that the system of equations (76)–(77), which also describes all particular features of the instantaneous state of the stochastic thermohydrodynamic fields of the turbulized flow of a gas–dust disk medium and their variations under given initial and boundary conditions, often cannot be solved using currently available computers. This is because using numerical methods entails the approximation of the colossal space-time field of turbulized flow parameters by a finite number of mesh points that should be used to solve the finite-difference approximations of the differential equations. At present, there is only one economically justified way out: to solve the hydrodynamic equations (76)–(77) only for large space-time scales of motion that determine the averaged structure parameters of such a stochastic medium and to model all of the smaller scales phenomenologically. In this case, stochasticity means the existence of an ensemble of possible realizations of pulsating gas-suspension flow fields for which the concept of statistically mean (mathematical expectation) is defined for all thermohydrodynamic parameters.

## AVERAGED EQUATIONS OF TWO-PHASE MECHANICS TO DESCRIBE THE TURBULENT HEAT AND MASS TRANSFER IN A GAS–DUST DISK

Before developing the basics of the phenomenological theory for the turbulence of multiphase media, as applied to the problem of modeling the evolution of the circumsolar protoplanetary disk, note once again the following: the currently available approaches to describing multiphase turbulent flows are imperfect (see, e.g., Shraiber *et al.*, 1987). This is attributable both to incompleteness of the “classical” theory of turbulence in “ordinary” hydromechanics and to a cardinal complication of the pattern of turbulent gas flow in the presence of a disperse admixture. It should be kept in mind that, being a fundamental problem of heterogeneous mechanics, the problem of the inverse effect of solid particles on the flow parameters has not yet been solved in full. In particular, this concerns the methods of allowance for the collective effects related to interparticle interactions whose role increases with particle concentration and size. For example, the mechanism of intense randomization of the motion of large particles (the so-called pseudoturbulence), which are weakly entrained by the turbulent pulsations of the carrier medium (see Shraiber *et al.*, 1980), is related to interparticle collisions. Thus, in view of the above peculiarities of turbulent flows in heterogeneous media, any theoretical approaches to their description and mathematical models based on them will always be limited, since they essentially pertain to a strictly definite range of concentrations and inertias of the disperse phase. This also applies to all of the currently existing models for the evolution of a gas–dust disk that cover a relatively narrow range of problems pertaining to the problem in question.

Let us now turn to deriving the basic balance equations of matter, momentum, and energy for a disk gas–dust turbulized medium intended to formulate and numerically solve the specific problems of consistently modeling the thermohydrodynamic parameters of a protoplanetary cloud at various stages of its evolution and analyze the physical meaning of the individual terms in these equations. For the functions in these equations to be smooth and continuous with continuous first derivatives, Eqs. (76) must be averaged over time or an ensemble. The progress made in recent years in developing and using semiempirical models for turbulence of the first order of closure (the so-called gradient models) for a homogeneous compressible fluid (see, e.g., Taunsend, 1959; Van Migem, 1977; Kolesnichenko and Marov, 1999) allows us to generalize some of these models to the shear flows of a two-phase gas–dust mixture that we describe in terms of a single-fluid continuum. We will derive the closing (defining) relations for the turbulent flows of phase diffusion, heat, and the Reynolds turbulent stress tensor by a standard method based on the concept of mixing length.

### *Choosing the Averaging Operator*

Various methods of averaging the fields of physical quantities are known to be used in the theories of fluid and gas turbulence. These include, for example, the time averaging

$$\overline{\mathcal{A}} \equiv (1/\Delta t)\int_{t-\Delta t/2}^{t+\Delta t/2}\mathcal{A}(\mathbf{x}, t)\mathrm{d}t, \qquad (78)$$

where the averaging interval $\Delta t$ of the pulsating parameter $\mathcal{A}(\mathbf{x}, t)$ is assumed to be large compared to the characteristic pulsation period and essentially small compared to the variation period of the averaged field $\overline{\mathcal{A}}(\mathbf{x}, t)$; the space averaging through integration over the space volume; the space-time averaging; the probability-theoretical averaging over the ensemble of possible realizations; etc. (see, e.g., Monin and Yaglom, 1992). The latter approach using the concept of an ensemble, i.e., an infinite set of stochastic hydrodynamic systems of the same nature that differ by the state of the field of velocities and/or other thermohydrodynamic parameters of motion at a given instant in time, is most fundamental. According to the well-known ergodicity hypothesis, the time and ensemble averages for a stationary stochastic process are identical. Without discussing the advantages and shortcomings of various averaging methods, we only note that the practice of constructing phenomenological models to study the turbulent motions shows that, in general, the methods of introducing the averaged parameters of motion are unimportant for setting up a complete system of averaged hydrodynamic equations if we require that the Reynolds postulates be satisfied during any averaging:

$$\overline{\mathcal{A}+\mathcal{B}} = \overline{\mathcal{A}}+\overline{\mathcal{B}}, \quad \overline{a\mathcal{A}} = a\overline{\mathcal{A}}, \quad \overline{\overline{\mathcal{A}}\mathcal{B}} = \overline{\mathcal{A}}\overline{\mathcal{B}}. \qquad (79)$$

Here, $\mathcal{A}(\mathbf{x}, t)$ and $\mathcal{B}(\mathbf{x}, t)$ are pulsating parameters of the turbulent field of physical parameters for the system; $\overline{\mathcal{A}}(\mathbf{x}, t)$ and $\overline{\mathcal{B}}(\mathbf{x}, t)$ are their mean values; and $a$ is a constant. We will also assume that any averaging operator used in (79) commutes with the differentiation and integration operators in both space and time[31]

$$\begin{cases} \overline{\partial\mathcal{A}(\mathbf{x}, t)/\partial t} = \partial\overline{\mathcal{A}(\mathbf{x}, t)}/\partial t \\ \overline{\int\mathcal{A}(\mathbf{x}, t)\mathrm{d}t} = \int\overline{\mathcal{A}(\mathbf{x}, t)}\mathrm{d}t \\ \overline{\partial\mathcal{A}(\mathbf{x}, t)/\partial x_j} = \partial\overline{\mathcal{A}(\mathbf{x}, t)}/\partial x_j \\ \overline{\int\mathcal{A}(\mathbf{x}, t)\mathrm{d}x_j} = \int\overline{\mathcal{A}(\mathbf{x}, t)}\mathrm{d}x_j. \end{cases} \qquad (80)$$

In the classical theories of the turbulence of homogeneous incompressible fluids that have been developed to date fairly completely (see, e.g., Monin and Yaglom, 1992), the averagings for all thermohydrodynamic parameters without exception are usually introduced in an identical way and, as a rule, without weighting coefficients. In the time averaging (78) or the probability-theoretical averaging over the ensemble of possible realizations

$$\overline{\mathcal{A}} \equiv \lim_{N\to\infty}\sum_{p-1}^{N}\mathcal{A}^{(p)}/N \qquad (81)$$

(where the summation is over the set of realizations, and the corresponding average field $\overline{\mathcal{A}}$ is defined as the mathematically expected value of $\mathcal{A}$ for an ensemble of identical systems), the actual value of the parameter $\mathcal{A}$ is represented as the sum of the averaged, $\overline{\mathcal{A}}$, and pulsational, $\mathcal{A}'$, components: $\mathcal{A} = \overline{\mathcal{A}} + \mathcal{A}'$ (with $\overline{\mathcal{A}'} = 0$). In this case, the separation of the actual stochastic motion into a slowly changing continuous mean motion and a rapidly oscillating turbulent (irregular, pulsating about the means) motion depends entirely on the choice of the space-time region for which the means are defined. The size of this region fixes the scale of mean

[31]Some of the relations (79)–(80) after the time (space) averaging hold only approximately, although they will be the more accurate, the smaller the change of the mean $\overline{\mathcal{A}}(\mathbf{x}, t)$ in time (and/or space) in the domain of integration under consideration.

motion.[32] All vortices of larger sizes contribute to the averaged motion determined by the averaged thermohydrodynamic parameters. All vortices of smaller sizes filtered out in the averaging process contribute to the small-scale turbulent motion determined by the corresponding pulsations of the same structure parameters.

At the same time, this averaging method (the same for all variables) for a two-phase continuum with a pulsating total density ρ leads not only to cumbersome hydrodynamic equations of mean motion, which is related to the need for retaining correlations of the type $\overline{\rho'\mathbf{u}'}$, $\overline{\rho'\mathbf{u}'\mathbf{u}'}$, $\overline{\rho' C'_\alpha \mathbf{u}'}$, etc. (which emerge, because the convective terms of the basic equations for instantaneous motion are nonlinear) in their structure, but also to difficulties in physically interpreting the individual terms of the averaged equations. Therefore, when developing our models for a gas-dust disk medium, apart from the "ordinary" means for some of the pulsating parameters, we will use below the so-called Favre (1969) weighted mean averaging for several other parameters specified, for example, by the relation

$$\begin{aligned}\langle \mathcal{A} \rangle &\equiv \overline{\rho\mathcal{A}}/\bar{\rho}\\ &= \left[\lim_{N\to\infty}\frac{1}{N}\sum_{p=1}^{N}\rho^{(p)}\mathcal{A}^{(p)}\right]\Big/\left[\lim_{N\to\infty}\frac{1}{N}\sum_{p=1}^{N}\rho^{(p)}\right];\end{aligned} \tag{82}$$

in this case, $\mathcal{A} = \langle\mathcal{A}\rangle + \mathcal{A}''$, $\overline{\mathcal{A}''} \neq 0$; here, $\mathcal{A}''$ is the corresponding turbulent pulsation. Thus, we will use two symbols to denote the mean quantities: the overbar denotes the ensemble (time and/or space) averaging, while the angular brackets denote the weighted mean averaging. The double prime is used below to denote the pulsations about the Favre averaged quantity. Note that the Favre averaging of several pulsating thermohydrodynamic parameters for a heterogeneous continuum simplifies significantly the form and analysis of the averaged hydrodynamic equations. This is because for the ordinary averaging, correlations of the type $\overline{\rho'\mathcal{A}'}$, $\overline{\rho'\mathcal{A}'\mathbf{u}'}$, etc. appear in the averaged equations of motion in explicit form, while for the Favre averaging, these correlations are hidden in the corresponding terms of the equations that have a simpler form.

Below, we list some of the properties of the weighted mean averaging widely used below that can be easily derived from definition (82) and the Reynolds relations (80) (see, e.g., Van Migem, 1977):

$$\begin{cases}\overline{\rho\mathcal{A}''} = 0, \quad \overline{\mathcal{A}''} = -\overline{\rho'\mathcal{A}''}/\bar{\rho}\\ \overline{\rho\mathcal{A}\mathcal{B}} = \bar{\rho}\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle + \overline{\rho\mathcal{A}''\mathcal{B}''}\\ \overline{\nabla\langle\mathcal{A}\rangle} = \nabla\langle\mathcal{A}\rangle\\ (\mathcal{A}\mathcal{B})'' = \langle\mathcal{A}\rangle\mathcal{B}'' + \langle\mathcal{B}\rangle\mathcal{A}'' + \mathcal{A}''\mathcal{B}'' - \overline{\rho\mathcal{A}''\mathcal{B}''}/\bar{\rho}\\ \overline{\rho d\mathcal{A}/dt} = \bar{\rho}D\langle\mathcal{A}\rangle/Dt + \nabla\cdot\overline{(\rho\mathcal{A}''\mathbf{u}'')},\end{cases} \tag{83}$$

where

$$D(\ldots)/Dt \equiv \partial(\ldots)/\partial t + \langle\mathbf{u}\rangle\cdot\nabla(\ldots) \tag{84}$$

is the substantial derivative for the averaged motion.

### *Averaged Mass Balance Equations*

Thus, we will consider the turbulized two-phase disk medium as a continuum whose instantaneous motions can be described by the system of hydrodynamic equations (76) for a random sample of initial and boundary conditions.[33] The macroscopic equations of turbulent motion for the gas–dust medium can then be obtained (in a form convenient for the subsequent analysis) by the stochastic ensemble averaging of Eqs. (76) using the weighted means for such flow parameters as the velocity $\langle\mathbf{u}\rangle$, temperature $\langle T\rangle$, mass concentrations $\langle C_\alpha\rangle$, etc. However, it is convenient to average the pressure $p$ and density ρ of the medium as well as all "molecular" thermodynamic fluxes $\mathbf{J}_\alpha$, $\mathbf{q}$, $\mathbf{\Pi}$, and $\xi_\rho$ in the "ordinary" way, i.e., without using any weighting coefficients.[34]

**The averaged continuity equation.** It is easy to see that the averaged density $\bar{\rho}$ and the weighted mean hydrodynamic velocity $\langle\mathbf{u}\rangle = \overline{\rho\mathbf{u}}/\bar{\rho}$ satisfy the continuity equation for mean motion

$$\begin{aligned}&\frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot(\bar{\rho}\langle\mathbf{u}\rangle) = 0,\\ \text{or}\quad &\bar{\rho}\frac{D}{Dt}\left(\frac{1}{\bar{\rho}}\right) - \nabla\cdot\langle\mathbf{u}\rangle = 0.\end{aligned} \tag{85}$$

Regarding this equation, it is important to emphasize the following: with known difficulties in modeling the binary correlations $\overline{\rho'\mathbf{u}'}$ that appear after the "ordinary" averaging of Eq. (13) for the true density and

[32]There can be various procedures of deriving the equations for large-scale turbulence: the "smoothed" thermohydrodynamic parameters can be introduced, in particular, using the filter function $\bar{\mathcal{A}}(\mathbf{x}, t) = \int G(\mathbf{x} - \mathbf{x}')\mathcal{A}(\mathbf{x}', t)d\mathbf{x}'$ (Leonard, 1974) or when setting up the (momentum, mass, etc.) balance equations for each cell of the computational mesh (Ievlev, 1970).

[33]This is possible for the space-time scales enclosed between the scales of molecular motions and the minimum turbulence scales (the linear size and lifetime of the smallest vortices), which are generally several (at least three) orders of magnitude larger than the scales of molecular motions, i.e., the separation between the molecules, let alone the sizes of the molecules (see, e.g., Van Migem, 1977).

[34]The Favre averaging of the equations of motion for a two-phase flow of a gas suspension described in terms of a single-fluid continuum was probably performed in this paper for the first time.

hydrodynamic velocity of a two-phase system without any weight, the preservation of the standard form (85) for the averaged continuity equation is a convincing argument for using the weighted mean $\langle \mathbf{u} \rangle$ for the total hydrodynamic flow velocity (see Kolesnichenko and Marov, 1999).

In particular, formula (83) widely used below can be derived only using Eq. (85) by averaging the operator relation $\rho d\mathcal{A}/dt = \partial(\rho\mathcal{A})/\partial t + \nabla \cdot (\rho\mathcal{A}\mathbf{u})$; as a result, we have

$$\overline{\rho\frac{d\mathcal{A}}{dt}} = \frac{\partial\overline{(\rho\mathcal{A})}}{\partial t} + \nabla \cdot (\overline{\rho\mathcal{A}\mathbf{u}})$$

$$= \frac{\partial\overline{(\rho\mathcal{A})}}{\partial t} + \nabla \cdot [\bar{\rho}\langle\mathcal{A}\rangle\langle\mathbf{u}\rangle + \overline{\rho\mathcal{A}''\mathbf{u}''}] \qquad (83^*)$$

$$= \bar{\rho}\frac{D}{Dt}\langle\mathcal{A}\rangle + \nabla \cdot \mathbf{J}^{turb}_{\mathcal{A}},$$

where we denote

$$\mathbf{J}^{turb}_{\mathcal{A}} \equiv \overline{\rho\mathcal{A}''\mathbf{u}''} = \bar{\rho}\langle\mathcal{A}''\mathbf{u}''\rangle \qquad (86)$$

for the second single-point moments of the flow velocity pulsations and some transferred substance $\mathcal{A}$. Thus, formula (86) introduces the so-called turbulent flux related to the transfer of substance $\mathcal{A}$ by the turbulent pulsations of the system's hydrodynamic velocity.

Below, we give a formula for the turbulent flux $\mathbf{J}^{turb}_{v}$ of the specific volume $v(\mathbf{x}, t)$ ($\equiv 1/\rho$). The flux $\mathbf{J}^{turb}_{v}$ plays an important role in our approach and appears in many averaged equations of motion, for example, in the averaged energy equation (see Eq. (126*)). Using the relation $v'' = -\rho'/\rho\bar{\rho}$, which directly follows from the definition of pulsations $v''$ ($v'' = v - \langle v\rangle = 1/\rho - 1/\bar{\rho} = -\rho'/\rho\bar{\rho}$), we obtain from (86)

$$\mathbf{J}^{turb}_{v} \equiv \overline{\rho v''\mathbf{u}''} = -\overline{\rho'\mathbf{u}''}/\bar{\rho} = \overline{\mathbf{u}''}. \qquad (87)$$

Below, we everywhere assume that only the dust volume content $s$ and the true gas density $\rho_g$ fluctuate in the gas–dust flow (this is a cardinal assumption of the approach developed here); it then follows from Eq. (2) that

$$\frac{\rho'}{\bar{\rho}} \cong (1-\bar{s})\frac{\rho'_g}{\bar{\rho}} + \langle\sigma\rangle s' = \langle C_g\rangle\frac{\rho'_g}{\bar{\rho}_g} + \langle\sigma\rangle s', \qquad (88)$$

where we denoted

$$\langle\sigma\rangle \equiv (\rho_d - \bar{\rho}_g)/\bar{\rho} \cong \rho_d/\bar{\rho} \qquad (89)$$

for the averaged excess of the dust particle density above the gas-suspension density and use the expression

$$\langle C_g\rangle \equiv \frac{\overline{(1-s)\rho_g}}{\bar{\rho}} \cong \frac{(1-\bar{s})\bar{\rho}_g}{\bar{\rho}} \cong \frac{\bar{\rho}_g}{\bar{\rho}} \qquad (90)$$

for the averaged mass concentration of the gas phase ($\langle C_d\rangle \equiv \rho_d\bar{s}/\bar{\rho} \cong \bar{s}\langle\sigma\rangle$, $\langle C_g\rangle + \langle C_d\rangle = 1$). The following expression for the turbulent flux of the specific volume $\mathbf{J}^{turb}_{v}$ in the gas–dust medium follows from (88) in (87):

$$\mathbf{J}^{turb}_{v} \equiv -\overline{\rho'\mathbf{u}''}/\bar{\rho} = -\langle\sigma\rangle\overline{s'\mathbf{u}''} - \left(1 - \frac{\rho_d\bar{s}}{\bar{\rho}}\right)\frac{\overline{\rho'_g\mathbf{u}''}}{\bar{\rho}_g}. \qquad (91)$$

It should be noted that Eqs. (88) and (90) for $\rho'$ and $\langle C_g\rangle$ (as many similar quantities that will appear below) are valid only when the inequalities $\overline{\mathcal{A}'\mathcal{B}'}/\bar{\mathcal{A}}\bar{\mathcal{B}} \ll 1$ and $\langle\mathcal{A}''\mathcal{B}''\rangle/\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle \ll 1$ hold for any pulsating thermodynamic parameters $\mathcal{A}$ and $\mathcal{B}$ not equal to the gas–dust flow velocity $\mathbf{u}$; below, we everywhere assume that the ratios of this kind are small without any special stipulations.

**The averaged diffusion equation for the disperse component of a disk system.** Applying the averaging operator (83*) to the diffusion equation (12) for disperse particles, we obtain the balance equation for the dust concentration

$$\bar{\rho}\frac{D\langle C_d\rangle}{Dt} + \nabla \cdot (\bar{\mathbf{J}}_d + \mathbf{J}^{turb}_d) = \overline{\sigma_{dg}},$$
$$\overline{\sigma_{dg}} \equiv \sum_{\rho=1}^{r} \nu_{d,\rho}\bar{\xi}_\rho. \qquad (92)$$

Here, $\langle C_d\rangle = \rho_d\bar{s}/\bar{\rho}$; $\bar{\mathbf{J}}_d$ is the averaged "molecular" diffusion flux of the dust defined by the relation (see formula (12))

$$\bar{\mathbf{J}}_d \equiv \overline{\rho C_d C_g \mathbf{w}} \cong \bar{\rho}\langle C_d\rangle\langle C_g\rangle\bar{\mathbf{w}} \cong \rho_d\frac{\bar{\rho}_g}{\bar{\rho}}\bar{s}\bar{\mathbf{w}}; \qquad (93)$$

$$\mathbf{J}^{turb}_d \equiv \overline{\rho C''_d\mathbf{u}''} = \rho_d\overline{s\mathbf{u}''} \qquad (94)$$

is the so-called turbulent diffusion flux of the disperse phase (for the gas diffusion flux, we may write $\mathbf{J}^{turb}_g \equiv \overline{\rho C''_g\mathbf{u}''} = -\overline{\rho C''_d\mathbf{u}''} = -\mathbf{J}^{turb}_d$).

If we write the turbulent dust flux as $\mathbf{J}^{turb}_d = \rho_d\bar{s}\,\mathbf{J}^{turb}_v + \rho_d\overline{s'\mathbf{u}''}$, then we can obtain the following representation for the turbulent flux of the specific volume using (91):

$$\mathbf{J}^{turb}_{v} = -\langle\sigma\rangle\frac{\bar{\rho}}{\rho_d\bar{\rho}_g}\mathbf{J}^{turb}_d - (1-\bar{s})\frac{\overline{\rho'_g\mathbf{u}''}}{\bar{\rho}_g}. \qquad (91^*)$$

Below, we will need this expression for $\mathbf{J}^{turb}_{v}$.

To close the averaged equation (92), we must have a defining relation for the turbulent diffusion flux of the dust, $\mathbf{J}^{turb}_d \equiv \bar{\rho}\langle C''_d\mathbf{u}''\rangle$. There are several approaches to modeling second-order correlation moments of this type that differ in complexity (see, e.g., Marov and Kolesnichenko, 2002). Here, we restrict our analysis to the simplest gradient relation that we will derive in a traditional way, by introducing the concept of mixing length.[35] For this purpose, we will assume that the transfer of particular flow field characteristics $\mathcal{A}$ by the turbulent pulsations of the medium takes place as a diffusion process and that the existence of an effective mixing length $\xi_{\mathcal{A}}$ of substance $\mathcal{A}$, the distance to which the turbulent moles (vortices) in the flow move before they are destroyed through the interaction with other disturbances, may be admitted. If we denote the Lagrangian turbulent pulsation of the transferred substance $\mathcal{A}$ corresponding to the Eulerian pulsation $\mathcal{A}''$ by $\mathcal{A}_L''$ and the effective mixing length by $\xi_{\mathcal{A}}$, then we may write $\mathcal{A}_L'' = \mathcal{A}'' + \xi_{\mathcal{A}}\nabla(\langle\mathcal{A}\rangle)$[36] $C_d'' = -\xi_d\nabla\langle C_d\rangle$. Hence, the diffusion flux $\mathbf{J}_d^T$ of the system's dust component in terms of the gradient representations is

$$\begin{aligned}\mathbf{J}_d^{turb} &\equiv \overline{\rho C_d''\mathbf{u}''} = -\bar{\rho}\langle\mathbf{u}''\xi_d\rangle\cdot\nabla\langle C_d\rangle \\ &= -\bar{\rho}\mathbf{D}_d^{turb}\cdot\nabla\langle C_d\rangle = -\bar{\rho}\rho_d\mathbf{D}_d^{turb}\cdot\nabla\left(\frac{\bar{s}}{\bar{\bar{\rho}}}\right),\end{aligned} \tag{95}$$

where the dyad $\mathbf{D}_d^{turb} \equiv \langle\mathbf{u}''\xi_d\rangle$ defines the nonsymmetric turbulent diffusivity tensor of the dust that, in the general anisotropic case, allows for the differences in the intensities of the turbulent pulsations of the solid particle velocity and concentration along different coordinate axes. Relation (95) is equivalent to the assertion that the turbulent flux of the dust phase is proportional to the gradient in mean concentration $\langle C_d\rangle$ and has the direction opposite to it. It is important to keep in mind that using the gradient hypothesis does not eliminate all of the difficulties associated with the closure problem, since the corresponding turbulent transfer coefficients must also be determined (experimentally or based on a qualitative physical analysis).

For an isotropic turbulent field, we may assume that the tensor $\mathbf{D}_d^{turb}$ is spherical, $\mathbf{D}_d^{turb} = \mathbf{I}D_d^{turb}$, i.e., is defined by the turbulent diffusivity of the dust $D_d^{turb}(\mathbf{x})$ alone (a statistical parameter of turbulence); then,

$$\begin{aligned}\mathbf{J}_d^{turb} &= -\bar{\rho}D_d^{turb}\nabla\langle C_d\rangle \\ &\equiv -D_d^{turb}\frac{\rho_d\bar{\rho}_g}{\bar{\rho}}[\nabla\bar{s} - \bar{s}\nabla\ln\bar{\rho}_g]\end{aligned} \tag{96}$$

and the averaged diffusion equation (92) takes the form

$$\bar{\rho}\frac{D\langle C_d\rangle}{Dt} + \nabla\cdot\left\{\bar{\rho}_g\langle C_d\rangle\bar{\mathbf{w}} - \frac{\bar{\rho}\nu^{turb}}{Sc^{turb}}\nabla\langle C_d\rangle\right\} = \bar{\sigma}_{dg}, \tag{92*}$$

where $\nu^{turb}$ is a turbulent analog of the coefficient of kinematic viscosity for a gas–dust mixture (see below); $Sc^{turb} \equiv \nu^{turb}/D_d^{turb}$ is the turbulent Schmidt number for the disperse phase (a dimensionless factor of the order of unity that depends on the nature of the dust substance and that is a function of the dimensionless flow parameters). In the gradient theory, the Schmidt number is calculated using the formula $Sc^{turb} = \xi_u/\xi_d$, where $\xi_u$ is the velocity mixing length. The dependence of the Schmidt number on the dust particle concentration was first obtained by Abramovich and Girshovich (1973).

**The turbulent transfer coefficient; the Stocks number.** First, note that the coefficients of turbulent transfer in any turbulized medium, in contrast to the corresponding molecular transfer coefficients, describe not just its thermophysical properties, but also the state of the turbulent field and, hence, depend on the averaging scale of the pulsating thermohydrodynamic parameters. For this reason, the way of introducing the averaged characteristics of turbulent motion is crucial in developing the methods for experimentally determining this kind of transfer coefficients. The assumption that the particles are completely entrained by the turbulent pulsations of the scale that plays a major role in particle encounter mechanics (the approximation of a passive admixture) underlies the most advanced approach to modeling the turbulent diffusivity. If the solid particles are very small and, hence, their motion does not differ in any way from the motion of the gas carrier moles, then the turbulent diffusivity of the dust particles $D_d^{turb}$ and the coefficient of turbulent viscosity $\nu^{turb}$ of the gas are equal for them. In this case, $D_d^{turb}$ depends only on the turbulent pulsation scale length of the carrier gas and can be estimated, for example, as

$$D_d^{turb} \sim \nu^{turb} = \sqrt{b}l \sim (\varepsilon l)^{1/3}l = \varepsilon^{1/3}l^{4/3}, \quad \text{when } l > \lambda_K. \tag{97}$$

The following notation is used in this expression: $b$ is the turbulent energy of the gas–dust medium as a whole (see (107)); $\varepsilon \cong b^{3/2}/l \cong \nu_g^3/\lambda_K^4$ is the dissipation rate of the turbulent gas energy (see Eq. (125*)); $\lambda_K \cong (\nu_g^3/\varepsilon)^{1/4}$ is the Kolmogorov (internal) turbulence scale length; and $l(\mathbf{x})$ is the Prandtl mixing length (a numerical factor that can be included in $l$). Below, we call $l$ the turbulence scale length at a given point in the flow.

It should be noted, however, that the numerous experimental data (see, e.g., Mednikov, 1981) confirm the equality $D_d^{turb} \cong \nu^{turb}$ only for very small particles, when the dimensionless Stokes number in the large-scale pulsational motion Syk $\ll 1$. In general, several Stokes numbers Stk, which are equal to the ratio of the dynamic relaxation time of the dust particles to particular flow time scales (e.g., to the Kolmogorov turbu-

[35]In the past decade, deeper (in physical content) differential turbulence models have come to be used to model turbulent single-phase flows in thin accretion disks. Apart from the equations for averaged quantities, these models include the additional differential transfer equations for the most important parameters of the turbulent structure (see, e.g., Dubrulle, 1992).

[36]The dust substance is assumed to be indestructible; that is, the dust amount in an elementary volume is not changed while it is moving without mixing with the ambient gas.

lence time scale $\tau_K \cong (\nu_g/\varepsilon)^{1/2}$ or to the time scale of the large-scale pulsational motion of the medium $\tau_L \propto b/\varepsilon$) that characterize the particle inertia relative to the chosen flow scale in the turbulent flow, can be introduced for a turbulized heterogeneous flow. In the case of Keplerian differential rotation of the solid particles in the disk, where there is a radial gradient in averaged velocity, it is important to take into account the particle inertia when analyzing the relaxation of the averaged phase velocities. For this purpose, it is convenient to introduce the Stokes number in the averaged motion, which we will write as $\mathrm{Stk} = \omega_{turb}\tau_{relax}$, where $\tau_{relax}$ is the dynamic relaxation (dynamic inertia) time of the particles; $\omega_{turb}$ is the lower frequency limit for the turbulent pulsations of the carrier gas in the disk belonging to the largest vortices with the (macroscopic turbulence) scale length $L$; the frequency $\omega_{turb}$ then determines the slow macroscopic variations in flow parameters (which are generally not related to turbulence) and, according to Safronov (1969), is specified in the form $\omega_{turb} = \Omega_{K,\,mid}$, where $\Omega_{K,\,mid} \equiv \sqrt{G\mathcal{M}_\odot/r^3}$ is the orbital frequency (the Keplerian angular velocity near the midplane of the disk). Cuzzi *et al.* (1993) slightly refined this estimate: $\omega_{turb} \approx \zeta\Omega_{K,\,mid}$, where $\zeta \approx 0.0126$.

For small spherical particles (e.g., with diameters $\ll 1$ cm at 1 AU or ~600 cm at 10 AU), the dynamic relaxation time scale is defined by the Epstein law (see Eq. (21))

$$\tau_{relax}^{Ep} = \frac{\tilde{\rho}_d\tilde{\rho}_g}{\rho R_{dg}} = \frac{\rho_d\tilde{d}_d}{2\rho c_{sg}}, \quad \tilde{d}_d < \lambda_g \tag{98}$$

(the mean free path of gas molecules at 1 AU is $\lambda_g \sim$ 1 cm). However, for coarse spherical particles, this formula is slightly modified. The simplest expression for $\tau_{relax}$ can be obtained when the Reynolds number for dust, $\mathrm{Re}_d = \tilde{d}_d|\mathbf{w}|/\nu_g = 2\tilde{d}_d|\mathbf{w}|/\lambda_g c_{sg}$, is fairly small, $\mathrm{Re}_d < 1$ (which is the case for the so-called Stokes particles). This inequality is true, for example, for particles with diameters from 1 to 10 cm at 1 AU and with diameters from 600 to 1000 cm at 10 AU (Dubrulle *et al.*, 1995). In this case, according to Eq. (22), the coefficient of aerodynamic resistance is $C_D(\mathrm{Re}_d) = 9\mathrm{Re}_d^{-1}$ and the dynamic relaxation time scale $\tau_{relax}$ will be defined by the Stokes law

$$\tau_{relax}^{St} = \frac{\tilde{\rho}_d\tilde{\rho}_g}{\bar{\rho}R_{dg}} = \frac{\tilde{d}_d\rho_d}{2\bar{\rho}C_D(\mathrm{Re}_d)|\mathbf{w}|}$$
$$\cong \frac{\tilde{d}_d\rho_d\mathrm{Re}_d}{18\bar{\rho}|\mathbf{w}|} = \frac{\rho_d\tilde{d}_d^2}{9\bar{\rho}c_{sg}\lambda_g} = \frac{\rho_d\tilde{d}_d^2}{18\bar{\rho}\nu_g}, \tag{99}$$
$$\tilde{d}_d > \lambda_g.$$

Thus, the inertia of a Stokes particle depends on the parameters of the medium in which it moves. In addition, if the particles are not too small (and, hence, are not entirely entrained by the gas carrier moles), then their relative velocities acquired through turbulent pulsations depend significantly on their masses. For the motion of a non-Stokes particle, its inertia depends on the Reynolds number for dust $\mathrm{Re}_d$ and may be written as

$$\tau_{relax} = \tau_{relax}^{St}/C(\mathrm{Re}_d), \tag{99*}$$

where

$$C(\mathrm{Re}_d) = \begin{cases} 1 + 0.179\mathrm{Re}_d^{1/2} + 0.013\mathrm{Re}_d, & \mathrm{Re}_d \le 10^3 \\ 0.0183\mathrm{Re}_d, & \mathrm{Re}_d > 10^3 \end{cases}$$

is a correction function that allows for the influence of inertial forces on the relaxation time scale of the non-Stokes particle (the coefficient of aerodynamic resistance for the particle is $C_D(\mathrm{Re}_d) = 9\mathrm{Re}_d^{-1}C(\mathrm{Re}_d)$). The difference between the pulsation velocities of particles of different sizes determines their encounter and increases the collision probability.[37] Thus, the following formula (cf, e.g., Cuzzi *et al.*, 1993) is valid for a polydisperse disk medium:

$$\mathrm{Sc}^{turb} = \frac{\nu^{turb}}{D_d^{turb}} \cong (1 + \mathrm{Stk})\sqrt{1 + 3|\bar{\mathbf{w}}|^2/2b},$$
$$\text{where} \quad \mathrm{Stk} = \zeta\Omega_{K,\,mid}\frac{\rho_d\tilde{d}_d^2}{18\rho\nu_g C(\mathrm{Re}_d)}. \tag{100}$$

**A defining equation for the averaged relative velocity.** Averaging the defining equation (19*) for the actual values of vector $\mathbf{w}$ yields

$$\bar{\rho}\theta_{gd}\bar{\mathbf{w}} \cong -\frac{D\bar{\mathbf{w}}}{Dt} - \overline{(\mathbf{w}\cdot\nabla)\mathbf{u}} + \frac{1}{\bar{\rho}_g}\nabla\bar{p}$$
$$= -\frac{D\bar{\mathbf{w}}}{Dt} - \mathbf{J}_v^{turb}\cdot\nabla\bar{\mathbf{w}} - \overline{\mathbf{u}''\cdot\nabla\mathbf{w}'} - (\bar{\mathbf{w}}\cdot\nabla)\langle\mathbf{u}\rangle$$
$$-\overline{(\mathbf{w}'\cdot\nabla)\mathbf{u}''} + \frac{1}{\bar{\rho}_g}\nabla\bar{p} \tag{101}$$
$$-(\bar{\mathbf{w}}\cdot\nabla)\mathbf{J}_v^{turb} = -\frac{D\bar{\mathbf{w}}}{Dt} - (\bar{\mathbf{w}}\cdot\nabla)\langle\mathbf{u}\rangle + \frac{1}{\bar{\rho}_g}\nabla\bar{p}.$$

In writing this relation, we disregarded the pulsations $\mathbf{w}'$ of the relative velocity (which is valid only when the velocity of the averaged relative phase motion $\bar{\mathbf{w}}$ is much higher than the pulsation velocity $\mathbf{w}'$, i.e., for fairly large particles) and the products of the averaged thermodynamic fluxes of various natures as terms of the second order of smallness. In addition, we used the identical transformation

$$\overline{d\mathcal{A}}/dt \equiv D\bar{\mathcal{A}}/Dt + \mathbf{J}_v^{turb}\cdot\nabla\bar{\mathcal{A}} + \overline{\mathbf{u}''\cdot\nabla\mathcal{A}'}, \tag{102}$$

which can be tentatively written as $d\mathcal{A}/dt = D\mathcal{A}/Dt + \mathbf{u}''\cdot\nabla\mathcal{A} = D\mathcal{A}/Dt + \mathbf{u}''\cdot\nabla\bar{\mathcal{A}} + \mathbf{u}''\cdot\nabla\mathcal{A}$ by averaging the substantial derivative $d\mathcal{A}/dt$.

[37]The inertial coagulation of particles in a turbulized flow is also related to this fact.

*The Averaged Smoluchowski Coagulation Equation*

Turbulence leads to two types of phenomena that affect the coagulation in a disperse system. First, the particles acquire an additional relative velocity under the effect of turbulent pulsations, which, in turn, changes the coagulation kernel $K(W, U)$ that characterizes the particle collision probability in the system (see, e.g., Voloshchuk, 1984). Here, so far we can talk about two coagulation-accelerating effects with certainty. The first effect is related to an increase in the trapping coefficient through turbulent mixing; as a result, the number of collisions between solid particles increases significantly compared to a laminar flow. The second effect is related to the presence of a shear in the turbulent flow velocity field that causes a change in the trapping conditions at $K(W, U)$ close to zero and increases the coagulation probability of small particles (see, e.g., Woods *et al.*, 1972).

The phenomena of the second type are related to the collective behavior of particles in a turbulized system. Turbulence increases the local nonuniformities in the distribution of coagulating particles to scales comparable to the mean separation between the particles, thereby giving rise to fluctuations in the particle size distribution function $f(U, \mathbf{x}, t)$ at macroscopic distances. From a physical point of view, since the coagulation equation (33) is nonlinear, this variation in concentration of particles with volume $U$ results in an acceleration of the coagulation in a region with an enhanced particle concentration and in its deceleration in a region with a reduced particle concentration, so, on average, this leads to a different coagulation rate than in the uniform case ($U$ = const) and contributes to the faster appearance of large particles.

This process can be described by the formal averaging of the coagulation equation (33)

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\overline{f(U)}}{\bar{\rho}}\right) + \nabla\cdot[\mathbf{J}_f^{\mathrm{turb}}(U) + \overline{f(U)}\langle C_{\mathrm{g}}\rangle\overline{\mathbf{w}}]$$

$$= \frac{1}{2}\int_0^U \overline{f(W)}\,\overline{f(U-W)}K(W, U-W)\mathrm{d}W$$

$$- \overline{f(U)}\int_0^\infty \overline{f(W)}K(W, U)\mathrm{d}W \qquad (103)$$

$$+ \frac{1}{2}\int_0^U \overline{f'(W)f'(U-W)}K(W, U-W)\mathrm{d}W$$

$$- \int_0^\infty \overline{f'(U)f'(W)}K(W, U)\mathrm{d}W,$$

where

$$\mathbf{J}_f^{\mathrm{turb}}(U) \equiv \overline{\rho(f/\rho)''\mathbf{u}''} = -\bar{\rho}D_U^{\mathrm{turb}}\nabla\left(\frac{\overline{f(U)}}{\bar{\rho}}\right) \qquad (104)$$

is the turbulent flux of the dust particles of volume $U$; $\langle C_{\mathrm{g}}\rangle = (1-\bar{s})\bar{\rho}_{\mathrm{g}}/\bar{\rho}$; $D_U^{\mathrm{turb}}$ is the turbulent diffusivity for the fraction-$U$ particles, the expression for which was derived, for example, by Schmitt *et al.* (1997). Equation (103) is not closed, since the function $\gamma(U, W, \mathbf{x}, t) \equiv \overline{f'(U, \mathbf{x}, t)f'(W, \mathbf{x}, t)}$ is undefined. The equation for $\gamma(U, W, \mathbf{x}, t)$ can be derived by multiplying the basic equation (33) by $f$ and by its subsequent stochastic averaging over the ensemble of possible realizations to give an equation that contains the mean of the product of three functions $f'$. This operation leads to an infinite chain of equations. The problem of closing the latter was solved only by introducing a particular hypothesis.

If we integrate Eq. (103) over $U$, then the equation for the averaged total number of disperse particles $\overline{N}_{\mathrm{d}}(\mathbf{x}, t)$ will take the form

$$\bar{\rho}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\overline{N}_{\mathrm{d}}}{\bar{\rho}}\right) = -\nabla\cdot\left(\mathbf{J}_{N_{\mathrm{d}}}^{\mathrm{turb}} + \frac{\overline{N}_{\mathrm{d}}(1-\bar{s})\bar{\rho}_{\mathrm{g}}}{\bar{\rho}}\overline{\mathbf{w}}\right)$$

$$-\frac{1}{2}\int_0^\infty\int_0^\infty K(W, U)\bar{f}(W, \mathbf{x}, t)\bar{f}(U, \mathbf{x}, t)\mathrm{d}W\mathrm{d}U \qquad (105)$$

$$-\frac{1}{2}\int_0^\infty\int_0^\infty K(W, U)\gamma(U, W, \mathbf{x}, t)\mathrm{d}W\mathrm{d}U + \sum_k\sum_{\rho=1}^r \nu_{\mathrm{d}(k),\rho}\bar{\xi}_\rho,$$

where

$$\mathbf{J}_{N_{\mathrm{d}}}^{\mathrm{turb}} = \int_U \mathbf{J}_f^{\mathrm{turb}}(U)\mathrm{d}U \qquad (106)$$

is the turbulent flux of the number of dust particles for which the representation $\mathbf{J}_{N_{\mathrm{d}}}^{\mathrm{turb}} = \overline{N_{\mathrm{d}}\mathbf{u}''} = \mathbf{J}_{\mathrm{d}}^{\mathrm{turb}}/\rho_{\mathrm{d}}\tilde{U}_{\mathrm{d}}$ is valid. Since the function $\gamma(U, W, \mathbf{x}, t)$ must be positively defined in view of its symmetry in $U$ and $W$, the coagulation in a turbulized medium with nonuniformly distributed particles will be faster. In conclusion, note that, despite its importance, the question of how the fluctuations affect the coagulation rate has not yet been fully developed and requires a solution.

*The Averaged Equation of Motion for a Gas–Dust Disk Medium*

Given Eq. (83*), the Favre averaging of the instantaneous equation of motion (35) for the gas–dust mixture considered as a single entity yields

$$\bar{\rho}\frac{\mathrm{D}\langle\mathbf{u}\rangle}{\mathrm{D}t} \equiv \frac{\partial}{\partial t}(\bar{\rho}\langle\mathbf{u}\rangle) + \nabla\cdot(\bar{\rho}\langle\mathbf{u}\rangle\langle\mathbf{u}\rangle)$$

$$= -\nabla\bar{p}_{\mathrm{sum}} + \nabla\cdot(\mathbf{R} + \overline{\mathbf{\Pi}}_{\mathrm{sum}} + \overline{\mathbf{\Pi}}_{\mathrm{rel}}) + \bar{\rho}\frac{G\mathcal{M}_\odot}{|\mathbf{r}|^3}\mathbf{r}, \qquad (107)$$

where

$$\mathbf{R} \equiv -\overline{\rho \mathbf{u}''\mathbf{u}''} = -\bar{\rho}\langle \mathbf{u}''\mathbf{u}''\rangle \quad (108)$$

is the so-called turbulent (Reynolds) stress tensor, which takes the following form in Cartesian coordinates:

$$R_{ij} \equiv -\overline{\rho u_i'' u_j''} = \begin{pmatrix} -\overline{\rho u_1''^2} & -\overline{\rho u_1'' u_2''} & -\overline{\rho u_1'' u_3''} \\ -\overline{\rho u_2'' u_1''} & -\overline{\rho u_2''^2} & -\overline{\rho u_2'' u_3''} \\ -\overline{\rho u_3'' u_1''} & -\overline{\rho u_3'' u_2''} & -\overline{\rho u_3''^2} \end{pmatrix}. \quad (109)$$

The Reynolds tensor is a symmetric tensor of the second rank and describes the turbulent stresses produced by the pulsations of the turbulent velocity field in the gas–dust continuum as a whole. It is well known (see, e.g., Monin and Yaglom, 1992) that in a developed turbulent single-phase flow, i.e., at large values of the global Reynolds number $\mathrm{Re}_{\mathrm{glob}} = Lu_0/\nu$ corresponding to large-scale motions (here, $u_0$ are typical changes in the velocity of the gas–dust mixture at distances of the order of the macroscopic turbulence scale length $L$; and $\nu$ is the effective coefficient of kinematic viscosity for the gas suspension), we may disregard the averaged viscous stress tensor of the medium $\overline{\mathbf{\Pi}}$ compared to the Reynolds stress tensor $\mathbf{R}$, except the thin regions of the so-called viscous sublayer that border the solid substrate (in our case, the thin layer of dust adjacent to the midplane of the disk is this substrate). This is also valid for a differentially rotating Keplerian protoplanetary disk with a typical Reynolds number $\mathrm{Re}_{\mathrm{glob}} \ge 10^{10}$, since the turbulent viscosity of its material is larger than the molecular one by 8 orders of magnitude or more, as follows from the observed distribution of angular momentum and mass in the Solar system and in numerous systems of young stars with disks (see, e.g., Richard and Zahn, 1999). However, it should be kept in mind that the aforesaid does not apply to the averaged relative stress tensor of the phases $\overline{\mathbf{\Pi}}_{\mathrm{rel}}$, whose effect on the two-phase flow of the disk medium can be comparable in order of magnitude to the Reynolds tensor $\mathbf{R}$. In particular, near the subdisk, where the concentration of dust particles of fairly large sizes (and, hence, $\overline{\mathbf{w}} \gg 0$) is significant, these shear stresses act particularly effectively, leading to additional flow turbulization, but in a volume comparable to the volume of the dust layer, i.e., small compared to the total disk volume (see, e.g., Goldreich and Ward, 1973).

Using the fact that the Lagrangian pulsations of the weighted mean gas–dust flow velocity are conservative, $(\mathbf{u}'')_L \cong 0$, it can be shown that the Reynolds tensor $\mathbf{R}$ (for an isotropic turbulent field) is related to the gradients of Favre averaged flow velocity $\nabla\langle\mathbf{u}\rangle$ by the following defining relation[38] (see, e.g., Kolesnichenko and Marov, 1999)

$$\mathbf{R} = -2/3\bar{\rho}b\mathbf{I} + 2\bar{\rho}\nu^{\mathrm{turb}}\mathring{\mathbf{D}}, \qquad \mathring{\mathbf{D}} \equiv \mathbf{D} - 1/3\mathbf{I}\nabla\cdot\langle\mathbf{u}\rangle \quad (110)$$

or

$$\mathbf{R} = -2/3\bar{\rho}b\mathbf{I} + \bar{\rho}\nu^{\mathrm{turb}}(\nabla\langle\mathbf{u}\rangle + (\nabla\langle\mathbf{u}\rangle)^{\mathrm{transp}}) - 2/3\bar{\rho}\nu^{\mathrm{turb}}\mathbf{I}\nabla\cdot\langle\mathbf{u}\rangle, \quad (111)$$

where $\mathbf{D} \equiv 1/2(\nabla\langle\mathbf{u}\rangle + (\nabla\langle\mathbf{u}\rangle)^{\mathrm{transp}})$ is the averaged deformation tensor; $\mathring{\mathbf{D}}$ is the averaged deformation rate tensor;[39] $\nu^{\mathrm{turb}}$ is the kinematic coefficient of turbulent viscosity for the gas–dust mixture. The possible anisotropy in the coefficients of turbulent viscosity $\nu^{\mathrm{turb}}$ in a differentially rotating gas–dust cloud (see, Safronov, 1969) was analyzed in detail by Kolesnichenko (2000).

Relation (110) includes a key parameter in the theory of turbulence,

$$b \equiv \overline{\rho|u''|^2}/2\bar{\rho}, \quad (112)$$

the averaged kinetic energy of the turbulent pulsations in the weighted mean velocity of the gas–dust continuum (turbulent energy); in general, the corresponding balance equation (see (145)) is needed to determine it. Since a continuous distribution of velocity pulsations $\mathbf{u}''$ at various frequencies $f$ (from the minimum ones determined by the viscous forces to the maximum ones determined by the boundary conditions for the flow) is produced in a developed turbulent flow, it is often convenient to represent dispersion (112) as the sum of the corresponding quantities pertaining to different frequencies,

$$b = \int_0^\infty b(f)\mathrm{d}f, \quad (113)$$

where $b(f)$ is the fraction of the turbulent energy of the gas–dust mixture that corresponds to the frequency band $\mathrm{d}f$ (the energy spectrum for $b$).

Under the above assumptions, the averaged relative stress tensor $\overline{\mathbf{\Pi}}_{\mathrm{rel}}$ in Eq. (107) (recall that the tensor $\mathbf{\Pi}_{\mathrm{rel}}$ arises from the inertial effects of the relative motion of

[38]The defining relations for the Reynolds stress tensor $\mathbf{R}$ (110) and the averaged viscous stress tensor $\overline{\mathbf{\Pi}}$ were derived by Marov and Kolesnichenko (2002) by the methods of nonequilibrium thermodynamics from the averaged Gibbs relation.

[39]In what follows, we retain the designations $\mathbf{D}$ and $\mathring{\mathbf{D}}$ for the averaged deformation and deformation rate tensors, which cannot not lead to ambiguity (cf. Eq. (40).

the coarse dust particle fraction and gas (see Eq. (37))) can be transformed to

$$\begin{aligned}\overline{\mathbf{\Pi}}_{\rm rel} &\equiv -\overline{\rho C_{\rm d} C_{\rm g} \mathbf{w}\mathbf{w}} \\ &\equiv -\overline{\rho C_{\rm d} C_{\rm g}}(\overline{\mathbf{w}\mathbf{w}} + \overline{\mathbf{w}'\mathbf{w}'}) - \overline{\mathbf{w}\mathbf{w}(\rho C_{\rm d} C_{\rm g})'} \\ &\equiv -(\overline{\mathbf{w}\mathbf{w}} + \overline{\mathbf{w}'\mathbf{w}'})(\bar{\rho}\langle C_{\rm d}\rangle\langle C_{\rm g}\rangle + \overline{\rho C''_{\rm d} C''_{\rm g}}) \\ &\quad - 2\overline{\mathbf{w}}\overline{\mathbf{w}'(\rho C_{\rm d} C_{\rm g})'} - \overline{\mathbf{w}'\mathbf{w}'(\rho C_{\rm d} C_{\rm g})'} \\ &\cong -\bar{s}\rho_{\rm d}\langle C_{\rm g}\rangle(\overline{\mathbf{w}\mathbf{w}} + \overline{\mathbf{w}'\mathbf{w}'}) \\ &\quad - 2\overline{\mathbf{w}}(\rho_{\rm d}\overline{\mathbf{w}'s'}\langle C_{\rm g}\rangle + \overline{\rho C''_{\rm g}\mathbf{w}'}\langle C_{\rm d}\rangle + \overline{\rho C''_{\rm d} C''_{\rm g}\mathbf{w}'}).\end{aligned} \quad (114)$$

For a two-phase turbulent flow, all correlations in relations (114) are usually disregarded and only the first term is retained (see, e.g., Zuev and Lepeshinskii, 1981; Kartushinskii, 1984):

$$\overline{\mathbf{\Pi}}_{\rm rel} \cong -\bar{s}\rho_{\rm d}\langle C_{\rm g}\rangle\overline{\mathbf{w}}\overline{\mathbf{w}}, \quad (115)$$

which is valid only if the averaged relative velocity of the phases $\overline{\mathbf{w}}$ is much higher than the pulsation velocity $\mathbf{w}'$, i.e., for fairly large solid particles. For lower-inertia fine and medium particles, the first two terms in (114) should generally be retained; then,

$$\overline{\mathbf{\Pi}}_{\rm rel} = -\bar{\rho}_{\rm g}\langle C_{\rm d}\rangle\overline{\mathbf{w}}\overline{\mathbf{w}} + \mathbf{R}_{\rm rel}, \quad (116)$$

where $\mathbf{R}_{\rm rel} \equiv -\bar{\rho}_{\rm g}\langle C_{\rm d}\rangle\overline{\mathbf{w}'\mathbf{w}'}$ is the additional Reynolds stress tensor attributable to the pulsations of the relative velocity field of the phases. In gradient models, two methods are used to determine the pair correlations of this type. According to the first method, the correlation moments of $\mathbf{R}_{\rm rel}$ for relatively small particles are expressed directly in terms of the Reynolds stresses $\mathbf{R}$ for the gas–dust continuum as a whole, i.e., $\mathbf{R}_{\rm rel} = \beta\mathbf{R}$, where $\beta$ is the coefficient of entrainment of disperse particles into the pulsational gas motion (see, e.g., Gavin *et al.*, 1984). The second method of determining the additional turbulent stresses $\mathbf{R}_{\rm rel}$ uses gradient relations of type (111) with $\overline{\mathbf{w}}$ substituted for the averaged velocities $\langle\mathbf{u}\rangle$ and with the determination of the corresponding coefficient of turbulent viscosity (see, e.g., Melville and Bray, 1979).

### *The Balance Equation for the Averaged Internal Energy of a Mixture*

We will obtain the averaged energy equation for the gas–dust disk system as a whole by averaging the energy equation (45) for instantaneous motion over the ensemble of possible realizations. As a result, we have

$$\begin{aligned}\bar{\rho}\frac{{\rm D}\langle H_{\rm sum}\rangle}{{\rm D}t} &= \overline{\frac{{\rm d}p_{\rm sum}}{{\rm d}t}} - \nabla\cdot(\mathbf{q}^{\rm turb}_{\rm sum} + \bar{\mathbf{q}}_{\rm sum}) \\ &\quad + \overline{\Phi}_u + \overline{R_{\rm gd}|\mathbf{w}|^2 - s\sigma\mathbf{w}\cdot\nabla p},\end{aligned} \quad (117)$$

where $\mathbf{q}^{\rm turb}_{\rm sum} = \mathbf{q}^{\rm turb} + \mathbf{q}^{\rm turb}_{\rm rad}$;

$$\mathbf{q}^{\rm turb} \equiv \overline{\rho H''\mathbf{u}''} \cong \langle c_p\rangle\overline{\rho T''\mathbf{u}''} + \sum_\alpha\langle h_\alpha\rangle\mathbf{J}^{\rm turb}_\alpha \quad (118)$$

is the turbulent heat flux arising from the correlation between the enthalpy, $H''$, and hydrodynamic velocity, $\mathbf{u}''$, pulsations;

$$\begin{aligned}\mathbf{q}^{\rm turb}_{\rm rad} &\equiv \overline{\rho H''_{\rm rad}\mathbf{u}''} \cong \langle c_{p,\,{\rm rad}}\rangle\overline{\rho T''\mathbf{u}''}, \\ \langle c_{p,\,{\rm rad}}\rangle &\equiv 16a\langle T\rangle^3/3\bar{\rho}\end{aligned} \quad (119)$$

is the turbulent radiative heat flux. The approximate relations for the thermal energy fluxes (118) and (119) are written up to the terms containing triple correlations. Formula (118) can be easily derived using the algebraic property (83) of the Favre averaging and the expression

$$\begin{aligned}H'' &= \sum_\alpha[h''_\alpha\langle C_\alpha\rangle + \langle h_\alpha\rangle C''_\alpha + (C''_\alpha h''_\alpha)''] \\ &= \langle c_p\rangle T'' + \sum_\alpha C''_\alpha\langle h_\alpha\rangle + (c''_p T'')''\end{aligned} \quad (120)$$

for the specific enthalpy pulsations of the disk material. Here, $h''_\alpha = c_{p\alpha}T''$ is the partial enthalpy pulsation for phase $\alpha$ ($c_{p\alpha}$ = const);

$$\langle c_p\rangle = \sum_\alpha c_{p\alpha}\langle C_\alpha\rangle, \quad c''_p = \sum_\alpha c_{p\alpha}C''_\alpha \quad (121)$$

are, respectively, the averaged and pulsation components of the specific heat of the mixture at constant pressure. The averaged values for the enthalpies of radiation and matter appearing in Eq. (117) are defined by the relations

$$\begin{aligned}\langle H_{\rm rad}\rangle &\cong 4/3a\langle T\rangle^4/\bar{\rho}, \\ \langle H\rangle &\cong \langle c_p\rangle\langle T\rangle + \sum_\alpha h^0_\alpha\langle C_\alpha\rangle\end{aligned} \quad (122)$$

that follow from (43) and (44).

It is convenient to represent the substantial derivative of the total pressure in Eq. (117) using Eq. (102) as

$$\begin{aligned}\overline{{\rm d}p_{\rm sum}/{\rm d}t} &\equiv {\rm D}\bar{p}_{\rm sum}/{\rm D}t + \mathbf{J}^{\rm turb}_v\cdot\nabla\bar{p}_{\rm sum} + \overline{\mathbf{u}''\cdot\nabla p'_{\rm sum}} \\ &= {\rm D}\bar{p}_{\rm sum}/{\rm D}t + \mathbf{J}^{\rm turb}_v\cdot\nabla\bar{p}_{\rm sum} \\ &\quad + \nabla\cdot(\overline{p'_{\rm sum}\mathbf{u}''}) - \overline{p'_{\rm sum}\nabla\cdot\mathbf{u}''}.\end{aligned} \quad (123)$$

In addition, $\overline{\Phi_u}$ can be transformed as

$$\begin{aligned}\overline{\Phi_u} &\equiv \overline{\mathbf{\Pi}}_{\rm sum} : \nabla\langle\mathbf{u}\rangle + \overline{\mathbf{\Pi}_{\rm sum} : \nabla\mathbf{u}''} \\ &= \overline{\mathbf{\Pi}}_{\rm sum} : \mathbf{D} + \bar{\rho}\langle\varepsilon_e\rangle,\end{aligned} \quad (124)$$

where

$$\bar{\rho}\langle\varepsilon_e\rangle \equiv \overline{\mathbf{\Pi}_{\rm sum} : \nabla\mathbf{u}''} \tag{125}$$

is the dissipation rate of the turbulent kinetic energy of the gas–dust mixture into heat under the effect of "molecular" viscosity (a second key parameter in the theory of turbulence). It can be shown (see Marov and Kolesnichenko, 2002) that the dissipative term (125) for developed turbulence is slightly simplified,

$$\begin{aligned}\bar{\rho}\langle\varepsilon_e\rangle &\cong \bar{\mathbf{\Pi}}_{\rm sum} : \nabla\mathbf{J}_v^{\rm turb} + \overline{\mathbf{\Pi}' : \nabla\mathbf{u}'}\\ &\approx \overline{\mathbf{\Pi}' : \nabla\mathbf{u}'} \equiv \bar{\rho}\varepsilon \geq 0;\end{aligned} \tag{125*}$$

note that $\varepsilon$ (the "true" dissipation of turbulent energy) is always positive. Substituting (123) and (125*) in (117) yields an averaged energy equation for the gas–dust mixture in the form

$$\begin{aligned}\bar{\rho}\frac{\mathrm{D}\langle H_{\rm sum}\rangle}{\mathrm{D}t} &= \frac{\mathrm{D}\bar{p}_{\rm sum}}{\mathrm{D}t} - \nabla\cdot(\mathbf{q}_{\rm sum}^{\rm turb} - \overline{p'_{\rm sum}\mathbf{u}''} + \bar{\mathbf{q}}_{\rm sum})\\ &+ \bar{\mathbf{\Pi}}_{\rm sum} : \mathbf{D} + \overline{R_{\rm gd}|\mathbf{w}|^2 - s\sigma\mathbf{w}\cdot\nabla p}\\ &+ \mathbf{J}_v^{\rm turb}\cdot\nabla\bar{p}_{\rm sum} - \overline{p'_{\rm sum}\nabla\cdot\mathbf{u}''} + \bar{\rho}\varepsilon.\end{aligned} \tag{126}$$

Defining relations for the turbulent heat fluxes are required to close Eq. (126); these relations derived in our monograph (Marov and Kolesnichenko, 2002) are

$$\begin{aligned}\mathbf{q}^{\rm turb} &= \overline{p'\mathbf{u}''} - \chi^{\rm turb}\left\{\nabla\langle T\rangle - \frac{\nabla\bar{p}}{\bar{\rho}\langle c_p\rangle}\right\}\\ &+ \sum_\alpha \langle h_\alpha\rangle\mathbf{J}_\alpha^{\rm turb},\end{aligned} \tag{127}$$

$$\mathbf{q}_{\rm rad}^{\rm turb} = \overline{p'_{\rm rad}\mathbf{u}''} - \chi_{\rm rad}\left\{\nabla\langle T\rangle - \frac{\nabla\bar{p}_{\rm rad}}{\bar{\rho}\langle c_{p,\,\rm rad}\rangle}\right\}, \tag{128}$$

where

$$\chi^{\rm turb} = \bar{\rho}\langle c_p\rangle\frac{\nu^{\rm turb}}{\mathrm{Sc}^{\rm turb}}, \quad \chi_{\rm rad} = \frac{4ac\langle T\rangle^3}{3\kappa\bar{\rho}} \tag{130}$$

are, respectively, the coefficient of turbulent heat conductivity for the gas–dust medium and the coefficient of turbulent radiative heat conductivity; $\langle c_p\rangle = [\bar{\rho}_{\rm g}(1-\bar{s}) + \rho_{\rm d}\bar{s}\,c_{p\rm d}]/\bar{\rho}$ is the averaged specific heat (at constant pressure) for the total continuum.[40] According to Eq. (127), there are two mechanisms of thermal energy transfer through the gas suspension:

(1) under the effect of the averaged temperature (more precisely, potential temperature) gradient

$$\theta \equiv \mathrm{const}\langle T\rangle/\bar{p}^{\langle\mathfrak{R}\rangle/\langle c_p\rangle} \tag{131}$$

since $\dfrac{\nabla\theta}{\theta} = \dfrac{1}{\langle T\rangle}\left(\nabla\langle T\rangle - \dfrac{\nabla\bar{p}}{\bar{\rho}\langle c_p\rangle}\right) \approx \dfrac{1}{\langle T\rangle}(\nabla_z\langle T\rangle + G_{\rm a})$, where $G_{\rm a} \equiv g_z/\langle c_p\rangle$ is the adiabatic temperature gradient in the gas–dust disk (see Eq. (177));

(2) by the turbulent diffusion fluxes $\mathbf{J}_\alpha^{\rm turb} = -\bar{\rho}D^{\rm turb}\nabla\langle C_\alpha\rangle$ (see (96)), with each particle of phase $\alpha$ transferring, on average, the thermal energy $\langle h_\alpha\rangle$ (since $\sum_\alpha \mathbf{J}_\alpha^{\rm turb} = 0$, $D_{\rm d}^{\rm turb} = D_{\rm g}^{\rm turb} = D^{\rm turb}$).

It should also be noted that the first terms in (127) and (128) do not act as the energy flux, because $\overline{p'_{\rm sum}\mathbf{u}''}$ drops out of the complete energy equation (126), and were retained in Eqs. (127) and (128) only for convenience.

Let us now write Eq. (127) in a form useful for modeling a turbulized gas suspension. Using (96) to transform (127) yields

$$\begin{aligned}\mathbf{q}^{\rm turb} &= \overline{p'\mathbf{u}''} - \chi^{\rm turb}\left(\nabla\langle T\rangle - \frac{\nabla\bar{p}}{\bar{\rho}\langle c_p\rangle}\right)\\ &- D^{\rm turb}\bar{\rho}\sum_\alpha\langle h_\alpha\rangle\nabla\langle C_\alpha\rangle\\ &= \overline{p'\mathbf{u}''} - \frac{\chi^{\rm turb}}{\langle c_p\rangle}\left(\nabla\langle H\rangle - \frac{\nabla\bar{p}}{\bar{\rho}}\right),\end{aligned} \tag{127*}$$

where we made the usual (for the theory of turbulence) assumption that the turbulent Lewis number is equal to unity, $\mathrm{Le}^{\rm turb} = \chi^{\rm turb}/\bar{\rho}\langle c_p\rangle D^{\rm turb} = 1$ (see Monin and Yaglom, 1992).

It is occasionally convenient to write Eq. (126) via the averaged total energy $\langle E_{\rm sum}\rangle$ of the matter and radiation. Using for this purpose the transformation

$$\begin{aligned}&\bar{\rho}\mathrm{D}\langle E_{\rm sum}\rangle/\mathrm{D}t + \bar{p}_{\rm sum}\nabla\cdot\langle\mathbf{u}\rangle\\ &= \rho\mathrm{D}\langle H_{\rm sum}\rangle/\mathrm{D}t - \mathrm{D}\bar{p}_{\rm sum}/\mathrm{D}t\end{aligned} \tag{132}$$

(which is a corollary of the relation $\langle H_{\rm sum}\rangle = \langle E_{\rm sum}\rangle + \bar{p}_{\rm sum}/\bar{\rho}$ and the averaged continuity equation (85)), we obtain for a developed turbulent flow

$$\begin{aligned}\bar{\rho}\frac{\mathrm{D}\langle E_{\rm sum}\rangle}{\mathrm{D}t} &+ \nabla\cdot(\mathbf{q}_{\rm sum}^{\rm turb} - \overline{p'_{\rm sum}\mathbf{u}''} + \bar{\mathbf{q}}_{\rm sum})\\ &= -\bar{p}_{\rm sum}\nabla\cdot\langle\mathbf{u}\rangle + \bar{\mathbf{\Pi}}_{\rm sum} : \mathbf{D}\\ &+ \overline{R_{\rm gd}|\mathbf{w}|^2 - s\sigma\mathbf{w}\cdot\nabla p} + \mathbf{J}^{\rm turb}\cdot\nabla\bar{p}_{\rm sum}\\ &- \overline{p'_{\rm sum}\nabla\cdot\mathbf{u}''} + \bar{\rho}\varepsilon.\end{aligned} \tag{126*}$$

[40] Below, we will assume that $\mathrm{Sc}^{\rm turb} = \mathrm{Pr}^{\rm turb}$ in the disk, since the turbulent diffusivity and the coefficient of turbulent heat conductivity in a turbulized mixture are commonly assumed to be equal ($\chi^{\rm turb}/\bar{\rho}\langle c_p\rangle = D^{\rm turb}$), which is equivalent to the equality of the mixing lengths for matter and heat.

*The Turbulent Flux of the Specific Volume in a Gas–Dust Medium*

Let us now derive the final relation for the turbulent flux of the specific volume $\mathbf{J}_v^{\text{turb}}$ (see Eq. (91*)). For its derivation, let us first find the expression for the turbulent density pulsations $\rho_g'$ in the gas component of the mixture; as the equation of state for the latter, we will take, as previously, the equation of state for a perfect multicomponent gas

$$p = p_g = k_B T \sum_k n_{g(k)} = \rho_g \mathcal{R}_g T, \tag{133}$$

where

$$\mathcal{R}_g = k_B \sum_k n_{g(k)}/\rho_g = k_B \sum_k Z_k \tag{134}$$

is the so-called gas component for the mixture of gases; $Z_k = n_{g(k)}/\rho_g$ is the specific (per unit mass of the gas continuum) number density of component $k$. Representing the actual values of $\mathcal{R}_g$ and $T$ as the sums of the averaged and pulsational values ($\mathcal{R}_g = \langle\mathcal{R}_g\rangle + \mathcal{R}_g''$, $T = \langle T\rangle + T''$), let us rewrite (133) as

$$\begin{aligned} p &= \langle\mathcal{R}_g\rangle\rho_g\langle T\rangle + \mathcal{R}_g''\rho_g\langle T\rangle + \langle\mathcal{R}_g\rangle\rho_g T'' + \mathcal{R}_g''\rho_g T'' \\ &\cong \langle\mathcal{R}_g\rangle\rho_g\langle T\rangle + k_B\rho_g\langle T\rangle\sum_{k=1}^n Z_k'' \\ &+ \langle\mathcal{R}_g\rangle\rho_g T'' + k_B\rho_g\sum_{k=1}^n (Z_k''T''). \end{aligned} \tag{135}$$

If we now apply the statistical averaging operator to (135), then we will obtain the averaged equation of state for the thermal gas-suspension pressure

$$\begin{aligned} \bar{p} &= \langle\mathcal{R}_g\rangle\bar{\rho}_g\langle T\rangle + k_B\bar{\rho}_g\sum_{k=1}^n \langle Z_k''T''\rangle \\ &\cong \langle\mathcal{R}_g\rangle\bar{\rho}_g\langle T\rangle \end{aligned} \tag{136}$$

(note that the pulsational term in the averaged equation of state (136) is usually discarded in the theory of turbulence), which we will use to eliminate the product $\langle\mathcal{R}_g\rangle\langle T\rangle$ from (135); as a result, we will have for the pulsations $\rho_g'$ (Marov and Kolesnichenko, 1987)

$$\frac{\rho_g'}{\bar{\rho}_g} = \frac{p'}{\bar{p}} - \frac{\rho_g T''}{\bar{\rho}_g\langle T\rangle} - \frac{k_B\langle T\rangle\rho_g}{\bar{p}}\sum_{k=1}^n Z_k''. \tag{137}$$

It is well known that the relative pressure pulsations for gas flows with small Mach numbers (Ma) may be disregarded almost always compared to the relative temperature pulsations. This hypothesis (Morkovin, 1961), which has been tested up to Ma = 5, is probably also valid for turbulent motions in thin accretion disks: the motion along the $r$ and $z$ directions is subsonic, while the rotation velocity $u_\varphi$ exceeds the speed of sound $c_{gs}$ (the thin-disk condition $h_{\text{disk}} \ll r$, together with the expression $h_{\text{disk}} \approx c_{gs}/\Omega_{K,\,\text{mid}}$ for the disk thickness, requires that $h_{\text{disk}}/r \approx c_{gs}/u_\varphi \ll 1$). Below, we will also assume that the mean mass of the gas component of the gas suspension does not fluctuate; therefore, $\sum_{k=1}^n Z_k'' = (n_g/\rho_g)'' = 0$. Using (137), the correlation term (containing the true gas density pulsations in Eq. (91*) can then be rewritten as

$$-\frac{\overline{\rho_g'\mathbf{u}''}}{\bar{\rho}_g} - \frac{\overline{\rho C_g T''\mathbf{u}''}}{\bar{\rho}_g\langle T\rangle} \cong \langle C_g\rangle\frac{\overline{\rho T''\mathbf{u}''}}{\bar{\rho}_g\langle T\rangle} = \frac{\langle T''\mathbf{u}''\rangle}{\langle T\rangle} \tag{138}$$

(here, as everywhere below, the terms with triple correlations were discarded). For the final transformation of this expression, we will use the defining relation (127) for the turbulent heat flux (118):

$$\begin{aligned} -\frac{\overline{\rho_g'\mathbf{u}''}}{\bar{\rho}_g} &\cong \frac{1}{\langle T\rangle}\overline{\rho T''\mathbf{u}''} \\ &\cong \frac{1}{\langle T\rangle\langle c_p\rangle}\left(\mathbf{q}^{\text{turb}} - \sum_\alpha \langle h_\alpha\rangle\mathbf{J}_\alpha^{\text{turb}}\right). \end{aligned} \tag{139}$$

Finally, substituting (96) and (139) in (91*) and using (130), we will obtain the following defining relation for the turbulent flux of the specific volume for a heterogeneous mixture:

$$\begin{aligned} \mathbf{J}_v^{\text{turb}} &= -\langle\sigma\rangle\frac{\bar{\rho}}{\rho_d\bar{\rho}_g}\mathbf{J}_d^{\text{turb}} \\ &+ \frac{1}{\langle T\rangle\langle c_p\rangle}\left(\mathbf{q}^{\text{turb}} - \sum_\alpha\langle h_\alpha\rangle\mathbf{J}_\alpha^{\text{turb}}\right) \\ &\cong \bar{\rho}\frac{\nu^{\text{turb}}}{\text{Sc}^{\text{turb}}}\left[\frac{\rho_d}{\bar{\rho}_g}\nabla\left(\frac{\bar{s}}{\bar{\rho}}\right) - \frac{1}{\langle T\rangle}\left(\nabla\langle T\rangle - \frac{\nabla\bar{p}}{\bar{\rho}\langle c_p\rangle}\right)\right]. \end{aligned} \tag{140}$$

*Energy Balance Equations*

In a turbulized flow of disk material, compared to its laminar analog, there is a great variety of possible transfer mechanisms (conversion rates) between the various forms of the energies of motion of solid particles and gas that contribute to the conserved averaged total energy. To interpret the individual terms of the energy balance most accurately, let us consider the complete system of energy equations for the averaged fields of pulsating thermohydrodynamic parameters for a gas–dust cloud, including the balance equation for the kinetic energy of turbulent pulsations.

**The balance equation for the averaged kinetic energy of a gas–dust flow.** Given Eq. (38) for the gravitational force, a scalar multiplication of the equation of motion (107) by the velocity $\langle\mathbf{u}\rangle$ after the necessary transformations yields the following substantial form of the equation of live forces for the averaged motion of disk material (the momentum theorem):

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}(|\langle\mathbf{u}\rangle|^2/2) + \nabla\cdot[(\mathbf{I}\bar{p}_{\mathrm{sum}} - \mathbf{R} - \overline{\mathbf{\Pi}^*_{\mathrm{sum}}})\langle\mathbf{u}\rangle] \tag{141}$$
$$= p_{\mathrm{sum}}\nabla\cdot\langle\mathbf{u}\rangle - (\mathbf{R} + \overline{\mathbf{\Pi}^*_{\mathrm{sum}}}) : \nabla\langle\mathbf{u}\rangle - \bar{\rho}\langle\mathbf{u}\rangle\cdot\nabla\langle\Psi\rangle.$$

Here, $-\nabla\langle\Psi\rangle = \mathbf{g} = G\mathcal{M}_{\odot}\mathbf{r}/|\mathbf{r}|^3$, $|\langle\mathbf{u}\rangle|^2/2$ is the specific kinetic energy of the averaged motion of disk material. Although Eq. (141) is of an energy nature, it is not the law of conservation of energy in a turbulized continuum: Eq. (141) describes the law of conversion of the kinetic energy of the averaged gas-suspension motion into the work of external bulk and surfaces forces and the work of internal forces (and conversely) without allowance for the irreversible conversion of the mechanical disk energy into thermal and other forms of energy.

Let us explain the physical meaning of the individual terms in Eq. (141): the quantity $\nabla\cdot(\bar{p}_{\mathrm{sum}}\langle\mathbf{u}\rangle)$ is related to the outflow of mechanical energy from a unit volume of the disk medium per unit time; the divergence $\nabla\cdot[(\mathbf{R} + \overline{\mathbf{\Pi}^*_{\mathrm{sum}}})\langle\mathbf{u}\rangle]$ is the rate with which the total surface stress $(\mathbf{R} + \overline{\mathbf{\Pi}^*_{\mathrm{sum}}})$ in the averaged moving "gas suspension plus radiation" system does the work in a unit volume; the quantity $\bar{p}_{\mathrm{sum}}\nabla\cdot\langle\mathbf{u}\rangle$ (>0 or <0) is related to the rate of inverse adiabatic transformation of the averaged internal energy (heat) $\langle E_{\mathrm{sum}}\rangle$ into the system's mechanical energy (see Eq. (126*)) and is the work done per unit time in a unit volume against the averaged total pressure $\bar{p}_{\mathrm{sum}}$ by the flow of a moving gas suspension; the sign of $\bar{p}_{\mathrm{sum}}\nabla\cdot\langle\mathbf{u}\rangle$ depends on whether the mixture flow will expand ($\nabla\cdot\langle\mathbf{u}\rangle > 0$) or contract ($\nabla\cdot\langle\mathbf{u}\rangle < 0$); the quantity $(\mathbf{R} + \overline{\mathbf{\Pi}^*_{\mathrm{sum}}}) : \nabla\langle\mathbf{u}\rangle$ is the total rate of irreversible transformation of the kinetic energy of mean motion into other forms of energy (see Eqs. (126*), (144), and (149)), with the energy dissipating under the effects of both "molecular" viscosity with the rates $\overline{\mathbf{\Pi}}_{\mathrm{sum}} : \nabla\langle\mathbf{u}\rangle$ and $\overline{\mathbf{\Pi}}_{\mathrm{rel}} : \nabla\langle\mathbf{u}\rangle$ and turbulent viscosity with the rate $\mathbf{R} : \nabla\langle\mathbf{u}\rangle$.

Adding Eq. (141) and the balance equation for the potential energy of disk material

$$\bar{\rho}\frac{\mathrm{D}\langle\Psi\rangle}{\mathrm{D}t} \equiv \frac{\partial}{\partial t}(\bar{\rho}\langle\Psi\rangle) \tag{142}$$
$$+ \nabla\cdot(\bar{\rho}\langle\mathbf{u}\rangle\langle\Psi\rangle) = \bar{\rho}\langle\mathbf{u}\rangle\cdot\nabla\langle\Psi\rangle$$

yields the following transfer equation for the averaged mechanical energy of a turbulized gas–dust flow:

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{|\langle\mathbf{u}\rangle|^2}{2} + \langle\Psi\rangle\right)$$
$$+ \nabla\cdot[(\mathbf{I}\bar{p}_{\mathrm{sum}} - \mathbf{R} - \overline{\mathbf{\Pi}}_{\mathrm{sum}} - \overline{\mathbf{\Pi}}_{\mathrm{rel}})\langle\mathbf{u}\rangle] \tag{143}$$
$$= \bar{p}_{\mathrm{sum}}\nabla\cdot\langle\mathbf{u}\rangle - (\mathbf{R} + \overline{\mathbf{\Pi}}_{\mathrm{sum}} + \overline{\mathbf{\Pi}}_{\mathrm{rel}}) : \nabla\mathbf{D}.$$

**The balance equation for the averaged kinetic energy of the relative phase motion.** Averaging Eq. (43) and disregarding the third-order correlation terms (and, thus, the turbulent kinetic energy of the interphase diffusion[41] and the products of the thermohydrodynamic fluxes (e.g., the terms $\overline{\mathbf{\Pi}}_{\mathrm{rel}} : \nabla\mathbf{J}_v^{\mathrm{turb}}$) in the averaged gas–dust continuum as quantities of the second order of smallness, we obtain

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}(\langle C_{\mathrm{d}}\rangle\langle C_{\mathrm{g}}\rangle|\overline{\mathbf{w}}|^2/2) \approx \bar{\rho}\langle C_{\mathrm{d}}\rangle\langle C_{\mathrm{g}}\rangle\frac{\mathrm{D}}{\mathrm{D}t}(|\overline{\mathbf{w}}|^2/2) \tag{144}$$
$$\cong \overline{-R_{\mathrm{gd}}|\mathbf{w}|^2 + s\sigma\mathbf{w}\cdot\nabla p} + \overline{\mathbf{\Pi}}_{\mathrm{rel}} : \mathbf{D} - \bar{\rho}\sigma_{\mathrm{rel}},$$

where

$$\bar{\rho}\sigma_{\mathrm{rel}} \equiv -\overline{\mathbf{\Pi}'_{\mathrm{rel}} : \nabla\mathbf{u}'}. \tag{145}$$

Here, the heat dissipation $\overline{\mathbf{\Pi}}_{\mathrm{rel}} : \mathbf{D}$ (the mean work done by the relative stress tensor on the averaged velocity gradient $\nabla\langle\mathbf{u}\rangle \neq 0$ due to the relative phase velocity shear in the orbital motion of the disk material) is related to the conversion rate of the averaged kinetic diffusion energy into the kinetic energy of the averaged motion of the gas mixture as a whole (cf. (141)); $\sigma_{\mathrm{rel}}$ is the additional generation source of turbulent energy $b$ related to the presence of medium and large inertial particles in the flow (see Eq. (149*)). It should be kept in mind that, according to Gore and Crowe (1989), the turbulent vortex wakes that destabilize the gas flow and transform the energy of the averaged relative motion into the high-frequency components of the turbulence energy spectrum are formed only behind large particles (at Reynolds numbers of the flow around the particles $\mathrm{Re}_{\mathrm{d}} > 400$). In contrast, small particles ($\mathrm{Re}_{\mathrm{d}} < 110$) predominantly suppress the turbulence energy, spending it on their own acceleration (i.e., their entrainment into the pulsational motion of a polydisperse flow); the laminarizing effect of the disperse phase on the flow increases with decreasing particle inertia. As regards

[41] We disregard the turbulent kinetic energy of the interphase diffusion as a quantity of the third order of smallness and, thus, do not consider the specific form of the additional dissipative term related to the presence of fine particles (see Danon *et al.*, 1977) in the turbulent energy transfer equation for the gas–dust medium (149) considered as a single entity. It is important to note that for a flow with medium and large particles, whose relaxation time is significant, the additional dissipation of turbulent energy will be negligible compared to the other terms in Eq. (149) (see, e.g., Varaksin, 2003).

medium-sized particles ($110 < \mathrm{Re}_d < 400$), they have a mixed effect on the disk turbulence.

**The turbulent energy balance.** Let us now consider the transfer equation for the turbulent energy $b = \overline{|\mathbf{u}''|^2/2}$ of the gas–dust material of an accretion disk. This fundamental (in the theory of turbulence) equation or some of its modifications underlie many present-day semiempirical theories of turbulence (see Monin and Yaglom, 1971; Kolesnichenko and Marov, 1999). Using the equation for $b$ in the case of a heterogeneous medium, we can, in particular, analyze the dynamical effect of the disperse phase on the intensity of turbulence in a gas–dust disk medium and develop a phenomenological method of modeling the coefficient of turbulent viscosity for the medium by taking into account the influence of the inverse effects of the dust transfer and "potential" temperature on the decay (maintenance) of shear turbulence in a protoplanetary cloud. The balance equation for $b$ can be derived by various methods (see Marov and Kolesnichenko, 2002), one of which we will use in the case of a two-phase medium considered here.

Let $\mathcal{A}(\mathbf{x}, t)$ be the actual value of a scalar quantity (in particular, this can be the vector components) whose substantial balance is $\rho d\mathcal{A}/dt = -\nabla \cdot \mathbf{J}_{\mathcal{A}} + \sigma_{\mathcal{A}}$, where $\mathbf{J}_{\mathcal{A}}$ and $\sigma_{\mathcal{A}}$ are, respectively, the substantial flux density vector and the volume density of source $\mathcal{A}$. For example, for the equation of motion (35),

$$\mathcal{A} \equiv \mathbf{u}, \quad \mathbf{J}_{\mathcal{A}} \equiv -\mathbf{\Pi}^*_{\mathrm{sum}}, \quad \sigma_{\mathcal{A}} \equiv -p_{\mathrm{sum}}\mathbf{I} + \rho\mathbf{g}. \tag{146}$$

It is easy to show (this requires multiplying the identity $d\mathcal{A}''/dt = d\mathcal{A}/dt - D\langle\mathcal{A}\rangle/Dt - \mathbf{u}'' \cdot \nabla\langle\mathcal{A}\rangle$ by $\rho\mathcal{A}''$ and the Reynolds averaging of the result) that the balance equation for the root-mean-square pulsation $\langle\mathcal{A}''^2\rangle$ has the following general form (see Kolesnichenko, 1995):

$$\underbrace{\bar{\rho}\frac{D\langle\mathcal{A}''^2/2\rangle}{Dt}}_{\text{Convection}} + \nabla \cdot \underbrace{(\overline{\rho\mathcal{A}''^2\mathbf{u}''/2 + \mathcal{A}''\mathbf{J}_{(A)j}})}_{\text{Diffusion}}$$
$$= \underbrace{-\mathbf{J}^{\mathrm{turb}}_A \cdot \nabla\langle\mathcal{A}\rangle}_{\text{Reproduction}} + \underbrace{\overline{\mathcal{A}''\sigma_{\mathcal{A}}}}_{\text{Redistribution}} - \underbrace{\bar{\rho}\langle\varepsilon_{\mathcal{A}}\rangle}_{\text{Dissipation}}, \tag{147}$$

where

$$\bar{\rho}\langle\varepsilon_{\mathcal{A}}\rangle \equiv -\overline{\mathbf{J}_{\mathcal{A}} \cdot \nabla\mathcal{A}''} \tag{148}$$

is the rate of scalar dissipation of the dispersion $\langle\mathcal{A}''^2\rangle$. The generalized transfer equation (147) contains the terms that reflect the influence of the following processes on the space-time distribution of the dispersion $\langle\mathcal{A}''^2\rangle$: convective transfer, diffusion, the dispersion formation through the energy transfer between the averaged and pulsational motions, redistribution (between the pulsational motions in various directions), and the dissipation of the turbulent characteristic $\langle\mathcal{A}''^2\rangle$ through the "molecular" transfer processes.

Let us now substitute (146) in (147) and (148); as a result, we obtain the following transfer equation for the turbulent energy of a gas–dust mixture:

$$\bar{\rho}\frac{Db}{Dt} + \nabla \cdot \mathbf{J}^{\mathrm{turb}}_b = \mathbf{R} : \mathbf{D} - \mathbf{J}^{\mathrm{turb}}_v \cdot \nabla\bar{p}_{\mathrm{sum}} + \overline{p'_{\mathrm{sum}}\nabla \cdot \mathbf{u}''} - \bar{\rho}\langle\varepsilon_b\rangle, \tag{149}$$

where

$$\begin{cases} \mathbf{J}^{\mathrm{turb}}_b \equiv \overline{\rho(|\mathbf{u}''|^2/2 + p'_{\mathrm{sum}}/\rho)\mathbf{u}''} - \overline{(\mathbf{\Pi}_{\mathrm{sum}} + \mathbf{\Pi}_{\mathrm{rel}}) \cdot \mathbf{u}''} \\ \bar{\rho}\langle\varepsilon_b\rangle \equiv \overline{(\mathbf{\Pi}_{\mathrm{sum}} + \mathbf{\Pi}_{\mathrm{rel}}) : \nabla\mathbf{u}''}. \end{cases} \tag{150}$$

The estimates of the individual terms in Eq. (149) obtained for developed turbulence in our monograph (Marov and Kolesnichenko, 2002) allow it to be slightly simplified:

$$\bar{\rho}\frac{D\overline{|\mathbf{u}''|^2/2}}{Dt} + \nabla \cdot \{\overline{\rho(|\mathbf{u}''|^2/2 + p'_{\mathrm{sum}})\mathbf{u}''} - (\mathbf{\Pi}^*_{\mathrm{sum}})' \cdot \mathbf{u}''\}$$
$$= \mathbf{R} : \mathbf{D} + \overline{p'_{\mathrm{sum}}\nabla \cdot \mathbf{u}'} - \mathbf{J}^{\mathrm{turb}}_v \cdot \nabla\bar{p}_{\mathrm{sum}} - \bar{\rho}\varepsilon + \bar{\rho}\sigma_{\mathrm{rel}}, \tag{149*}$$

where

$$\bar{\rho}\langle\varepsilon_b\rangle = (\overline{\mathbf{\Pi}}_{\mathrm{sum}} + \overline{\mathbf{\Pi}}_{\mathrm{rel}}) : \nabla\mathbf{J}^{\mathrm{turb}}_v + \overline{\mathbf{\Pi}'_{\mathrm{sum}} : \nabla\mathbf{u}'} + \overline{\mathbf{\Pi}'_{\mathrm{rel}} : \nabla\mathbf{u}'} \cong \bar{\rho}\varepsilon - \bar{\rho}\sigma_{\mathrm{rel}}.$$

The first term on the left-hand side of Eq. (149*) describes the change with time (and the convective transfer by the averaged motion) in the kinetic energy of disk turbulence $b$, and the second term reflects the turbulent pulsation energy transfer through the turbulent "diffusion" processes; the quantity (energy dissipation)

$$\mathbf{R} : \mathbf{D} \equiv -2/3\bar{\rho}b\nabla \cdot \langle\mathbf{u}\rangle + 2\bar{\rho}\nu^{\mathrm{turb}}\mathring{\mathbf{D}} : \mathring{\mathbf{D}} \tag{150*}$$

(see Eq. (B.11)) on the right-hand sides of Eqs. (141) and (149*) appear with different signs and, therefore, it may be interpreted as the conversion rate of the kinetic energy of the averaged motion into the turbulence energy of the gas–dust disk medium considered as a whole[42] (this hydrodynamic generation mechanism of turbulence in a differentially rotating Keplerian disk is considered in this paper as the main mechanism (see Fridman, 1989)); the quantity $\overline{p'_{\mathrm{sum}}\nabla \cdot \mathbf{u}'}$ is related to the transformation rate of the internal energy of the gas suspension into the kinetic energy of turbulent vortices and is the work done per unit time in a unit volume of

[42]It should be emphasized that this energy conversion is a purely kinematic process that depends only on the chosen space-time averaging scale of the turbulent motion. For small-scale turbulence, the quantity $\mathbf{R} : \mathbf{D}$ is always positive, so small-scale turbulence always transforms the kinetic energy of the averaged motion into the kinetic energy of turbulent pulsations (this is the so-called dissipative effect of small-scale turbulence). At the same time, the kinetic energy of turbulence can be transferred to the energy of the averaged motion by large-scale turbulent vortices (see, e.g., Van Migem, 1977).

the pulsating medium on the vortices as a result of the existence of pulsations in the total pressure $p'_{\rm sum}$ of the disk medium and the expansion or contraction of turbulent vortices ($\nabla \cdot \mathbf{u}' > 0$ or $\nabla \cdot \mathbf{u}' < 0$); the quantity $\mathbf{J}_{v}^{\rm turb} \cdot \nabla \bar{p}_{\rm sum}$ is the conversion rate (in a unit volume of the medium) between the turbulent and averaged internal energies of the disk; small-scale vortices transform the turbulence energy into heat, since $\mathbf{J}_{v}^{\rm turb} \cdot \nabla \bar{p}_{\rm sum} > 0$ for them, while large vortex structures related to thermal convection (for which $\mathbf{J}_{v}^{\rm turb} \cdot \nabla \bar{p}_{\rm sum} < 0$ (see Kolesnichenko and Marov, 1999)) transform the thermal energy of the gas–dust flow into the averaged kinetic energy of the velocity pulsations (it should be noted that this generation mechanism of turbulence in the disk suggested by Lin and Papaloizou (1980) as the main mechanism cannot be considered in this capacity, since it is not universal and temporary in nature (see Ruden and Polack, 1991; Nomura, 2002)); the pair correlation $\bar{\rho}\,\varepsilon \equiv \overline{\mathbf{\Pi}'_{\rm sum} : \nabla \mathbf{u}'} > 0$ in a developed turbulent flow (the mean work done by the pulsations of the viscous stress tensor[43] on the turbulent vortices with a pulsation velocity gradient, $\nabla \mathbf{u}' \neq 0$) is the dissipation rate of the turbulent kinetic energy into heat under the effect of molecular viscosity (see Eq. (126*)); finally, the quantity $\bar{\rho}\,\sigma_{\rm rel} = -\overline{\mathbf{\Pi}'_{\rm rel} : \nabla \mathbf{u}'} > 0$, the work in the turbulent flow (per unit time in a unit volume) done by the pulsations of the relative stress tensor on the turbulent vortices (see Eq. 145)), may be interpreted as the additional generation of turbulence in the gas–dust disk arising from the inertial effects of the relative motion of the disperse and gas phases and related to the formation of a vortex wake behind large particles with sizes >1 cm (see, e.g., Zaichik and Varaksin, 1999). It is with this mechanism of flow turbulization by large-scale particles of centimeter sizes or larger (produced through coagulation and settling near the midplane of the protoplanetary cloud) that part of the additional source of turbulization of the gas-suspension flow near the thin dust layer (see Goldreich and Ward, 1973) that, in the opinion of many researchers largely, prevents further settling of small (micron-size) dust particles to the subdisk and, thus, delays the onset of direct gravitational instability of this layer[44] can be associated (see, e.g., Safronov, 1969; Weidenschilling, 1984; Goodman and Pindor, 2000).

The correlation $\bar{\rho}\,\sigma_{\rm rel}$, up to the triple correlation terms, can be transformed to

$$
\begin{aligned}
\bar{\rho}\sigma_{\rm rel} &= \overline{(s\rho_d C_g \mathbf{w}\mathbf{w})' : \nabla \mathbf{u}'} \\
&= \rho_d \overline{\mathbf{w}}\,\overline{\mathbf{w}}\,\overline{s C_g : \nabla \mathbf{u}'} + 2\rho_d \overline{\mathbf{w}}\,\overline{s C_g \mathbf{w}' : \nabla \mathbf{u}'} \\
&\quad + \rho_d \overline{s C_g \mathbf{w}'\mathbf{w}' : \nabla \mathbf{u}'} \\
&\cong \rho_d \overline{\mathbf{w}}\,\overline{\mathbf{w}} : (\langle C_g \rangle \overline{s' \nabla \mathbf{u}'} + \bar{s} \overline{C'_g \nabla \mathbf{u}'}) \\
&\quad + 2\rho_d \bar{s} \langle C_g \rangle \overline{\mathbf{w}}\,\overline{\mathbf{w}' : \nabla \mathbf{u}'}.
\end{aligned}
\tag{151}
$$

We see from this relation that the additional generation of turbulence in the dusty disk (in particular, in the subdisk where relatively large solid particles are present) can arise from the averaged dynamical sliding of the phases, the correlation of the pulsations of the dust particle volume content and the gas concentration of the mixture with the flow pulsation velocity, and the pulsational interphase sliding. As we have repeatedly emphasized above, this additional source of turbulization is small for fine particles, for which the inertia effects may be ignored ($\mathbf{\Pi}_{\rm rel} \cong 0$).

To conclude this section, note that, in general, the energy equation (126*) for a gas–dust disk system is written in astrophysical literature by assuming a steady-nonequilibrium state of the turbulent field, where an internal equilibrium exists in the turbulence structure at which the production of turbulent entropy $S^{\rm turb}$ of the gas–dust material is approximately equal to its dissipation.[45] If we take this condition for the balance of $S^{\rm turb}$, then we will have (see Eq. (166))

$$
\begin{aligned}
&2\bar{\rho}\nu^{\rm turb} \overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} + \overline{p'_{\rm sum} \nabla \cdot \mathbf{u}'} \\
&- \mathbf{J}_{v}^{\rm turb} \cdot \nabla \bar{p}_{\rm sum} - \bar{\rho}\varepsilon + \bar{\rho}\sigma_{\rm rel} \approx 0,
\end{aligned}
\tag{152}
$$

and Eqs. (126*) for a developed turbulent flow can be transformed to

$$
\begin{aligned}
&\bar{\rho}\frac{D\langle E_{\rm sum}\rangle}{Dt} + \nabla \cdot (\mathbf{q}_{\rm sum}^{\rm turb} - \overline{p'_{\rm sum}\mathbf{u}''}) \\
&\cong -\bar{p}_{\rm sum} \nabla \cdot \langle \mathbf{u} \rangle + 2(\bar{\rho}\nu^{\rm turb} + \mu_{\rm rad}) \overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} \\
&\quad + \bar{\rho}\sigma_{\rm rel} + \overline{R_{gd}|\mathbf{w}|^2 - s\sigma \mathbf{w} \cdot \nabla p}
\end{aligned}
\tag{153}
$$

or, using (144),

$$
\begin{aligned}
&\bar{\rho}\frac{D}{Dt}(\langle E_{\rm sum}\rangle + \langle C_d \rangle \langle C_g \rangle |\mathbf{w}|^2/2) \\
&\cong -\nabla \cdot (\mathbf{q}^{\rm turb} - \overline{p'_{\rm sum}\mathbf{u}''} + \mathbf{q}_{\rm rad}^{\rm turb}) - \bar{p}_{\rm sum} \nabla \cdot \langle \mathbf{u} \rangle \\
&\quad + 2(\bar{\rho}\nu^{\rm turb} + \mu_{\rm rad}) \overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} + \overline{\mathbf{\Pi}}_{\rm rel} : \mathbf{D}.
\end{aligned}
\tag{154}
$$

[43] Recall that the value of the pulsations $\mathbf{\Pi}'$ of the viscous stress tensor is determined by the effective (including the disperse admixture) coefficient of kinematic viscosity for the gas–dust medium considered as a whole; thus, the greater the degree of entrainment of fine particles into the pulsational motion, the stronger their effect on the pulsational component of the tensor $\mathbf{\Pi}$ that produces additional dissipation of the turbulent gas-suspension energy.

[44] To reach a critical density in the dust layer requires a very high degree of its stabilization and flattening (Safronov, 1969).

[45] Unfortunately, when modeling a disk, some of the authors use the laminar energy equation with the coefficient of turbulent viscosity substituted for the coefficient of molecular viscosity without taking into account all the subtleties of deriving the energy equation for a turbulized gas suspension.

Thus, the equation for the internal energy of the averaged turbulized gas–dust continuum written via the absolute temperature takes the form

$$\bar{\rho}\langle c_p\rangle\frac{\mathrm{D}\langle T\rangle}{\mathrm{D}t}$$

$$-\nabla\cdot\left\{\chi^{\mathrm{turb}}\left(\nabla\langle T\rangle-\frac{\nabla\bar{p}}{\bar{\rho}\langle c_p\rangle}\right)+\chi_{\mathrm{rad}}\nabla\langle T\rangle\right\}$$
$$=\frac{\mathrm{D}\bar{p}_{\mathrm{sum}}}{\mathrm{D}t}+2(\bar{\rho}\nu^{\mathrm{turb}}+\mu_{\mathrm{rad}})\mathring{\mathbf{D}}:\mathring{\mathbf{D}} \tag{153*}$$
$$+\bar{\mathbf{\Pi}}_{\mathrm{rel}}:\mathbf{D}-\sum_{s=1}^{r}\langle q_s\rangle\bar{\xi}_s+\bar{\rho}\sigma_{\mathrm{rel}}+R_{\mathrm{gd}}|\bar{\mathbf{w}}|^2.$$

Here, we disregarded the energy of the dissipation $\overline{\Phi_u}$ (in a unit volume per unit time) through the "molecular" viscosity of the gas–dust mixture compared to the "frictional heat" $2\bar{\rho}\,\nu^{\mathrm{turb}}\mathring{\mathbf{D}}:\mathring{\mathbf{D}}$ through the viscous Reynolds stresses arising from the relative shear of the gas–suspension elements in the orbital motion of the disk material and the kinetic diffusion energy compared to the internal energy of the gas suspension. It should be noted that the presence of an internal heating source of the protoplanetary disk related to turbulence viscosity is in satisfactory agreement with the currently available astrophysical data[46] and with all cosmochemical constraints. In addition, under the assumptions made, an additional heating source of the gas–dust medium related to the energy dissipation under the effect of "molecular" diffusion $\bar{\mathbf{\Pi}}_{\mathrm{rel}}:\mathbf{D}$ appears in Eq. (153*); it plays an important role in the subdisk, where the relative velocities of the phases can be significant.

**The law of conservation of total energy for a turbulized mixture.** Adding the balance equations for the internal energy (126*), mechanical energy (143), kinetic energy of the interphase diffusion (144), and turbulent energy of the disk system (149) yields the law of conservation of total averaged energy for a two-phase gas–dust mixture and radiation[47] in the disk in a substantial form:

$$\bar{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\langle U_{\mathrm{tot}}\rangle = \nabla\cdot(\bar{\mathbf{J}}_U+\mathbf{J}_U^{\mathrm{turb}}), \tag{155}$$

where

$$\langle U_{\mathrm{tot}}\rangle = \langle E_{\mathrm{sum}}\rangle+\langle\Psi\rangle+\frac{1}{2}|\langle\mathbf{u}\rangle|^2$$
$$+\langle C_{\mathrm{d}}\rangle\langle C_{\mathrm{g}}\rangle|\bar{\mathbf{w}}|^2/2+\overline{|\mathbf{u}''|^2/2} \tag{156}$$

is the averaged total energy of the gas suspension and radiation (see Eq. (42));

$$\bar{\mathbf{J}}_U\equiv\overline{\mathbf{q}_{\mathrm{sum}}+(\mathbf{I}p_{\mathrm{sum}}-\mathbf{\Pi}_{\mathrm{sum}}-\mathbf{\Pi}_{\mathrm{rel}})\cdot\mathbf{u}}$$
$$\cong\bar{\mathbf{q}}_{\mathrm{sum}}+(\mathbf{I}\bar{p}_{\mathrm{sum}}-\bar{\mathbf{\Pi}}_{\mathrm{sum}}-\bar{\mathbf{\Pi}}_{\mathrm{rel}})\cdot\langle\mathbf{u}\rangle \tag{157}$$
$$+\overline{p_{\mathrm{sum}}\mathbf{u}''}-\overline{\mathbf{\Pi}'_{\mathrm{sum}}\cdot\mathbf{u}''}-\overline{\mathbf{\Pi}'_{\mathrm{rel}}\cdot\mathbf{u}''}$$

is the averaged actual total energy flux in the two-phase gas suspension;

$$\mathbf{J}_U^{\mathrm{turb}}\equiv\bar{\rho}\langle U''_{\mathrm{tot}}\mathbf{u}''\rangle$$
$$=\overline{\rho\left(H_{\mathrm{sum}}-p_{\mathrm{sum}}/\rho+\Psi+\frac{1}{2}|\mathbf{u}|^2+C_{\mathrm{d}}C_{\mathrm{g}}|\mathbf{w}|^2/2\right)\mathbf{u}''} \tag{158}$$
$$=\mathbf{q}_{\mathrm{sum}}^{\mathrm{turb}}-\overline{p_{\mathrm{sum}}\mathbf{u}''}+\overline{\rho|\mathbf{u}''|^2\mathbf{u}''/2}-\mathbf{R}\cdot\langle\mathbf{u}\rangle+\mathbf{q}_{\mathrm{sum}}^{\mathrm{turb}}$$

is the turbulent total energy flux of the mixture.

Combining Eqs. (157) and (158), we obtain the following expression for the total energy flux of the turbulized flow of a gas–dust mixture:

$$\bar{\mathbf{J}}_U+\mathbf{J}_U^{\mathrm{turb}} = \overline{\rho|\mathbf{u}''|^2\mathbf{u}''/2}-\overline{(\mathbf{\Pi}'_{\mathrm{sum}}+\mathbf{\Pi}'_{\mathrm{rel}})\cdot\mathbf{u}''}$$
$$+\mathbf{q}_{\mathrm{sum}}^{\mathrm{turb}}+\bar{\mathbf{q}}_{\mathrm{sum}}+(\mathbf{I}\bar{p}_{\mathrm{sum}}-\bar{\mathbf{\Pi}}_{\mathrm{sum}}-\bar{\mathbf{\Pi}}_{\mathrm{rel}}-\mathbf{R})\cdot\langle\mathbf{u}\rangle. \tag{159}$$

Here, $\bar{\mathbf{q}}_{\mathrm{sum}}+\mathbf{q}_{\mathrm{sum}}^{\mathrm{turb}}$ is the total heat flux attributable to both the averaged molecular transfer and the turbulent transfer; $\bar{p}_{\mathrm{sum}}\langle\mathbf{u}\rangle$ is the "mechanical" energy flux; $(\bar{\mathbf{\Pi}}_{\mathrm{sum}}+\bar{\mathbf{\Pi}}_{\mathrm{rel}}+\mathbf{R})\cdot\langle\mathbf{u}\rangle$ is the total energy flux attributable to the work of viscous, relative, and turbulent stresses; $\overline{\rho|\mathbf{u}''|^2\mathbf{u}''/2}-\overline{(\mathbf{\Pi}'_{\mathrm{sum}}+\mathbf{\Pi}'_{\mathrm{rel}})\cdot\mathbf{u}''}$ is the "diffusion" flux of vortex turbulent energy. Note that the term $\overline{p_{\mathrm{sum}}\mathbf{u}''}$ in (158) and (159) does not act as the energy flux, because it drops out of the total energy flux (159).

## MODELING THE COEFFICIENT OF TURBULENT VISCOSITY IN A GAS–DUST DISK

Let us now consider a semiempirical method for determining the coefficient of kinematic turbulent viscosity $\nu^{\mathrm{turb}}$ in a two-phase disk medium that includes the influence of the inertial effects of medium-size and coarse particles on the additional generation of turbulence in a gas–dust cloud. Below, the velocity shear of the cosmic material (when the kinetic energy of turbulence is extracted from the kinetic energy of averaged motion) related to its differential pattern of its rotation

[46]In the steady state, this heat released inside the disk due to viscosity is not accumulated, but is transferred to its surface (mainly through radiation) and is then radiated outward from the upper and lower disk surfaces.

[47]In this section, we retained the designation $U_{\mathrm{tot}}$ for the total energy of the "matter plus radiation" system (cf. Eq. (42)).

around the proton-Sun (see, e.g., Dubrulle, 1993; Gor'kavyi and Fridman, 1994) is assumed to be the main source of turbulence in the disk. Each layer of material with radius $r$ of a differentially rotating thin disk[48] ($h_{disk}(r) \ll r$) that lies near the $r\varphi$ plane (located at $z = 0$ in cylindrical coordinates) moves almost exactly according to Kepler's third law, i.e., it rotates increasingly fast as the central body (with mass $\mathcal{M}_{\odot}$) is approached: the Keplerian orbital velocity is $u_{\varphi}(r) = r\Omega_{K,\,mid}(r) = \sqrt{G\mathcal{M}_{\odot}/r}$ and the angular velocity of orbital rotation $\Omega_{K,\,mid}(r)$ increases as $r^{-3/2}$. Such motion is a typical case of shear flow, which can be analyzed in terms of the invariant modeling of developed turbulent flows in inhomogeneous media developed in our monograph (Kolesnichenko and Marov, 1999).

### *α-Parametrization of the Viscosity of a Protoplanetary Disk*

The coefficient of turbulent viscosity in an astrophysical gas-phase disk was first modeled by Shakura and Sunyaev (1973) in their now classical paper. Using Kolmogorov's concept of dynamic turbulent viscosity $\mu_g^{turb} = \bar{\rho}_g u_g^{turb} l_g^{turb}$ (where $u_g^{turb}$ is the root-mean-square velocity of turbulent pulsations limited by the speed of sound in the gas calculated in the midplane of the disk, $u_g^{turb} \le c_{sg}|_{z=0} \cong \sqrt{\bar{p}_g/\bar{\rho}_g}\big|_{z=0}$; $l_g^{turb}$ is the so-called Prandtl mixing length limited by the disk half-thickness $h_{disk}$, $l_g^{turb} \le h_{disk} \approx c_{sg}|_{z=0}/\Omega_{K,\,mid}$, $\rho_g$ and $p_g$ are, respectively, the mass density and pressure in the gas-phase disk), these authors obtained the relation

$$R_{r\varphi} = \bar{\rho}_g \nu_g^{turb}(r) r \partial_r (u_{\varphi}/r)$$
$$= -3/2\bar{\rho}_g u_g^{turb} l_g^{turb} \Omega_{K,\,mid}(r) = -\alpha \bar{p}_g\big|_{z=0}, \quad (160)$$
$$\alpha \le 1$$

between the $r,\varphi$ component of the Reynolds turbulent stress tensor $R_{r\varphi}$ and the thermal gas pressure $p_g$. The Shakura–Sunyaev disk parameter (a dimensionless free parameter) $\alpha$, which characterizes the degree of excitation of turbulent motions, can be calibrated empirically using time-dependent spectra obtained, in particular, during observations of outbursts in binary systems with mass transfer containing dwarf novae. For this case, it was found that $0.01 \le \alpha \le 1$ (see, e.g., Eardley *et al.*, 1978). The models of turbulized accretion disks constructed using relation (160) pertain to the so-called viscous $\alpha$-disks. Determining the parameter $\alpha$ under various assumptions about the nature of the physical processes in the disk was the subject of many studies (see, e.g., extensive bibliography to the paper by Makalkin (2004)). In particular, a number of authors (see Dubrulle, 1993; Cabot *et al.*, 1987), who used the $\alpha$-model when considering such physical mechanisms of turbulence in a protoplanetary disk as differential rotation, thermal convection, etc., obtained $\alpha \sim 10^{-3}$, the value that satisfies best the astrophysical constraints.

**Critical remarks.** The main advantage of such an heuristic approach to describing the disk turbulence is that it is relatively simple: it will suffice to substitute the coefficient of turbulent viscosity $\nu^{turb}(r)$ for the coefficient of molecular viscosity $\nu$ in the equations of stellar hydrodynamics to somehow take into account the turbulization of the medium in an accretion disk (in fact, this is what most astrophysicists do by essentially ignoring almost all correlation terms in the averaged equations of motion). At the same time, it is important to keep in mind that the Shakura–Sunyaev approach, which was specially developed by the authors to model thin (vertically uniform, i.e., structureless) astrophysical disks and which disregards the height dependence of the coefficient of turbulent viscosity, is appropriate to use only for global (one-dimensional in $r$) modeling of the evolution of the solar protoplanetary disk with parameters averaged over its thickness. In recent years, however, this approach has come to be used uncritically in astrophysical literature and in two-dimensional ($r$, $z$) models that are associated in one way or another with modeling of particular features of the vertical disk structure, in particular, with calculations of the height distribution of thermohydrodynamic parameters in the dusty subdisk, which, of course, is wrong.

In addition, it should be borne in mind that formula (160), which was derived for gas-phase disks, naturally disregards the inverse effect of dust and heat transfer on the development of turbulence in the disk,[49] which should be done when modeling many phenomena important for cosmogony. For example, when accurately modeling the evolution of the protoplanetary cloud as a viscous gas–dust disk that surrounded the Sun at an early stage of its existence, it is important to take into account the dynamical processes of gas–dust interaction and, in particular, the inverse effect of the inertial properties of dust particles on the intensity of turbulence and the thermal regime of the subdisk. The following is an argument for such a general approach: the dust particles, which account for only ~2% of the mass of the circumsolar protoplanetary cloud, may be disregarded only at the initial evolutionary stage of the cosmic system under consideration, when almost all of the primordial (interstellar) dust particles evaporated. At later stages of its evolution, as the protoplanetary cloud cooled down, the solid particles condensed, their sizes increased significantly through coagulation, the disperse phase settled to the midplane of the disk, and the gas dissipated from the disk system into interstellar

[48] Note that the disk thickness is not constant, but increases with distance from the proton-Sun (to a first approximation, $h_{disk}(r) \propto r$).

[49] This effect is that additional buoyant forces facilitating or preventing the growth of turbulence in the disk arise due to the difference between the concentrations of dust material mixed with the medium (during turbulent diffusion) or the temperature difference (during heat transfer) at separate points of the disk medium.

space, the dynamical, energetic, and optical roles of the dust component of the gas suspension increased significantly (see, e.g., Cuzzi *et al.*, 1993). At first glance, turbulent mixing hinders the diffusive separation of the dust and gas components in the gravitational field of the proto-Sun, preventing the settling of fine solid particles to its equatorial plane (where they form a flattened dust layer), and thus, inhibits the formation of a critical mass of the subdisk at which it becomes gravitationally unstable (Safronov, 1969). However, on the other hand, as we have repeatedly noted above, nongravitational accretion related, in particular, to the growth of particles through various turbulent coagulation mechanisms becomes an efficient accumulation mechanism of medium- and large-scale solid particles for turbulent flow. In addition, turbulence facilitates the formation of mesoscale, relatively stable gas–dust coherent structures that provide the most favorable conditions for the adhesion of dust particles. In such vortex structures, the number of collisions (per unit time) increases significantly, while the relative collision velocities decrease significantly compared to laminar conditions (through the combined coherent mesoscale motion of particles and small-scale turbulent pulsations of their relative velocities inside vortex structures), which also contributes to the growth of a condensed subdisk component (see Barge and Sommeria, 1995; Tanga *et al.*, 1996; Chavanis, 1999; Kolesnichenko, 2004). As the inertia of solid particles increases, they are drawn into the pulsational motion of the gas carrier flow to a progressively lesser extent. Thus, turbulence eventually contributes to an efficient settling of dust particles to the midplane of the disk and, thus, to the formation of a critical mass of the disk whose gravitational instability and disintegration lead to the formation of planetesimals.

The pattern of the disperse-phase effect on the dynamics of turbulent gas-suspension flow is not unique, but depends significantly on the inertia and volume content (concentration) of dust particles, since they can both laminarize and turbulize the flow (see Shraiber *et al.*, 1987). Kolesnichenko (2000) investigated the rotation-generated flows of disk material where the solid particles of the gas suspension begin to have an inverse effect on the parameters of the latter. Formula (160) for the coefficient of turbulent viscosity was generalized to the case of allowance for the low-inertia dust component where the approximation of a passive admixture (in which the two-phase gas–dust flow is approximated by the flow of a single-phase "multicomponent" medium with known effective thermophysical properties) could be used. It was recommended to apply the derived corrections to the coefficient of turbulent viscosity that incorporate the inverse effect of the transfer of a fine admixture and heat on the growth of turbulence when modeling the formation of a flattened dust layer in the disk.

At the same time, the influence of medium-size and coarse particles on the processes of turbulent heat and mass transfer in a protoplanetary gas–dust disk and their contribution to the correction to the coefficient of turbulent viscosity of a gas suspension remain an open question. Determining this kind of correction is a challenging problem and requires an in-depth study of the gas-suspension turbulence structure. In the next section, we attempt to theoretically determine the coefficient of turbulent viscosity $\nu^{turb}$ in a gas-suspension flow with large inertial dust particles.

### *Modeling the Coefficient of Turbulent Viscosity in a Dusty Subdisk*

Before determining the above correction to $\nu^{turb}$, we will recall that the relationship between the coefficient of turbulent viscosity and the gas-suspension turbulence energy is defined by Kolmogorov's relation (see formula (97) in Kolmogorov (1942))

$$\nu^{turb} = \gamma^* l\sqrt{b}, \tag{161}$$

where $l = l(\mathbf{x})$ is the turbulence scale length at a given point of the flow (the numerical factor $\gamma^*$ may be included in $l$). For a turbulized shear flow around an infinite plane (in our case, the equatorial plane of the disk, $z = 0$), the local turbulence scale length $l(\mathbf{x})$ may be assumed to be proportional to the thickness of the thin layer under consideration,

$$l(z) = \gamma^* \kappa z \tag{162}$$

or

$$l(z) = \gamma^* \kappa z \Phi(\mathrm{Re}_{glob}, \mathrm{Ri}, \mathrm{K}), \tag{163}$$

where $\Phi$ is a dimensionless function; $\kappa$ is the Karman constant, which may be set equal to ~0.4.

It should be kept in mind that deriving an adequate differential equation for the scale length $l(\mathbf{x})$ is one of the most complex problems in the semiempirical theory of shear turbulence. The point is that, in general, $l(\mathbf{x})$ cannot be defined only via the one-point moments of the pulsating velocity. Being a measure of the distance between two points, $\mathbf{x}_1$ and $\mathbf{x}_2$, in a turbulized flow at which the two-point correlators $\langle \mathbf{u}''(\mathbf{x}_1)\mathbf{u}''(\mathbf{x}_2)\rangle$ are still differ markedly from zero, the scale length $l(\mathbf{x})$ can be found from the complex differential equations for these moments by their integration over the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$ (see, for example, (Ievlev, 1975)). At the same time, even in the case of a given differential equation for $l(\mathbf{x})$, there is a complex problem of boundary conditions at the free boundary of the turbulent flow region, where the scale $l(\mathbf{x})$ does not tend to zero (see Laihtman, 1970). For these reasons, to ensure the efficiency of practical calculations, the local turbulence scale length $l(\mathbf{x})$ is often specified in the form of purely empirically found functions or calculated using an algebraic formula of type (163) that includes only the flow geometry, the distance to the wall, etc. and that does not depend on the peculiarities of the fluid flow.[50]

[50]The Prandtl–Nikuradze formula for $l(z)$ can probably be used in certain cases; for the plane case considered here, it may be written as $l(z) = \gamma 0.4z[1 - 1.1(z/h_{disk}) + 0.6(z/h_{disk})^2 - 0.15(z/h_{disk})^3]$.

**Internal equilibrium in the disk turbulence structure.** Let us now derive the sought-for correction to $\nu^{turb}(\mathbf{x})$. We will restrict our analysis to a simplified statistical scheme of turbulence in a two-phase medium based on the turbulent energy transfer equation (149*) alone (a one-parameter turbulence model).[51] If we disregard the small terms in Eq. (149*) (all of the necessary estimates can be found, for example, in our monograph (Marov and Kolesnichenko, 2002)), then, given Eq. (110) for the Reynolds tensor and Eq. (140) for a turbulent flow of specific volume $\mathbf{J}_v^{turb}$, it can be rewritten as

$$\bar{\rho}\left(\frac{\partial b}{\partial t} + \langle \mathbf{u}\rangle \cdot \nabla b\right) + \nabla \cdot \mathbf{J}_b^{turb} \cong -2/3 b\bar{\rho}\nabla \cdot \langle \mathbf{u}\rangle$$

$$+ 2\bar{\rho}\nu T\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} + \langle\sigma\rangle\frac{\bar{\rho}}{\rho_d\bar{\rho}_g}\mathbf{J}_{dz}^{turb} g_z \qquad (164)$$

$$+ \frac{g_z}{\langle T\rangle\langle c_p\rangle}\left(\mathbf{q}^{turb} - \sum_\alpha \langle h_\alpha\rangle \mathbf{J}_\alpha^{turb}\right)_z - \bar{\rho}\varepsilon + \bar{\rho}\sigma_{rel},$$

where $g_z$ is the vertical gravity of the proto-Sun (see Eq. (177)). As we already emphasized above, the contribution of additional dissipation to $\varepsilon$ is significant only for relatively fine dust particles; as a result, it plays no prominent role in the balance equation (164) for coarse dust.[52]

To write Eq. (164) in the form required for our subsequent purposes, we will use the basic concept of a two-level macroscopic description of a turbulized medium in the form of two interconnected continua (open subsystems) that simultaneously fill the same volume of coordinate space of the disk continuously, the subsystem of averaged motion and the subsystem of turbulent chaos, developed by Kolesnichenko (1998).[53] For the disk material, the continuum of averaged motion obtained through the probability-theoretical averaging of the instantaneous equations of motion for a heterogeneous medium (76) is intended to study the evolution of the averaged fields of thermohydrodynamic gas-suspension parameters (including also the possible large vortex structures). The subsystem of turbulent chaos for the disk (a vortex continuum with an internal structure) generally consists of two components: the turbulent chaos proper (the so-called incoherent turbulence) associated with the stochastic small-scale pulsational motion of an eddy fluid and the coherent component[54] embedded in this almost uniform pulsating field, an ensemble of mesoscale ordered vortex structures (multimolecular formations).

For the subsystem of turbulent chaos, we postulate the Gibbs fundamental identity (Kolesnichenko, 1998)

$$\delta E^{turb} = T^{turb}\delta S^{turb} - p^{turb}\delta(1/\bar{\rho}). \qquad (165)$$

Using this identity, we can determine in a well-know way (see, e.g., de Groot and Mazur, 1964) the thermodynamic structure of the vortex continuum, i.e., introduce the specific internal energy $E^{turb}(\mathbf{x}, t)$, specific entropy $S^{turb}(\mathbf{x}, t)$, pressure $p^{turb}(\mathbf{x}, t) \equiv 2/3\bar{\rho}E^{turb}$, and temperature[55] $T^{turb}(\mathbf{x}, t)$ of turbulization. The various relations between the parameters $E^{turb}$, $S^{turb}$, $T^{turb}$, and $p^{turb}$ that can be derived in a standard way may then be interpreted as the "equations of state" for the subsystem under consideration. Below, we assume that the internal energy of the subsystem of turbulent chaos $E^{turb}(\mathbf{x}, t)$ is identical to the turbulence energy, $E^{turb}(\mathbf{x}, t) = \overline{\rho|\mathbf{u}''|^2}/2\bar{\rho} = b(\mathbf{x}, t)$, and that the subsystem of turbulent chaos is an ideal gas in the thermodynamic sense, $\bar{\rho}b = 3/2p^{turb} = 3/2\mathcal{R}_{gd}\bar{\rho}T^{turb}$, where $\mathcal{R}_{gd} = k_B/\mathcal{M}_{gd}$, $\mathcal{M}_{gd}$ is the mean molecular mass of the gas-suspension particles (the cardinal assumptions of the model). Using the Gibbs identity (165), Eq. (164) takes the form

$$\bar{\rho}T^{turb}\frac{DS^{turb}}{Dt} + \nabla\mathbf{J}_b^{turb} \cong 2\bar{\rho}\nu^{turb}\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} - \frac{1}{\bar{\rho}_g}J_{dz}^{turb}g_z$$
$$+ \frac{g_z}{\langle T\rangle\langle c_p\rangle}\left(\mathbf{q}^{turb} - \sum_\alpha \langle h_\alpha\rangle\mathbf{J}_\alpha^{turb}\right)_z - \bar{\rho}\varepsilon + \bar{\rho}\sigma_{rel}, \qquad (166)$$

which, by analogy with (51), can be written in the form of a balance equation,

$$\frac{\partial}{\partial t}(\bar{\rho}S^{turb}) + \nabla \cdot \left(\bar{\rho}S^{turb}\langle\mathbf{u}\rangle + \frac{\mathbf{J}_b^{turb}}{T^{turb}}\right)$$
$$= \sigma_{S^{turb}} \equiv \sigma^e_{S^{turb}} + \sigma^i_{S^{turb}}.$$

Let us now use Eq. (166) to analyze the steady-nonequilibrium regime of developed ($\mathrm{Re}_{glob} \gg 1$) turbulence of the gas–dust mixture in a disk system. Naturally, its realization requires the existence of a continu-

[51] For two-phase flows, the development of a two-parameter ($b$–$\varepsilon$) turbulence model was initiated by Elghobashi and Abou-Arab (1982, 1983).

[52] This contribution was included in the dissipative term $2\bar{\rho}\nu^T\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}}$.

[53] Using the concept of a two-level macroscopic description of a turbulized fluid was the starting point that allowed us to phenomenologically develop a hydrodynamic model of structured turbulence as a self-organization in open nonequilibrium systems associated with fluctuating media (see Kolesnichenko, 2004).

[54] According to the currently available experimental data, the coherent turbulent structure can be defined as a connected, fluid mass with vorticity correlated in phase (i.e., coherent) in the entire region of coordinate space occupied by the structure. The formation of granules in the solar photosphere is a clear example of the existence of a vast family of coherent structures in a turbulent flow that appear against the background of small-scale turbulent motion.

[55] The thermodynamical temperature of turbulization of the subsystem of turbulent chaos is not reduced to the absolute temperature in the general case.

ously acting turbulization mechanism (e.g., the large-scale velocity shear of the flow of material related to its differential rotation around the proto-Sun or the thermal-convective large-scale instability) that transfers the kinetic energy of the averaged flow to the vertex motion on large scales and that does not allow the subsystem of turbulent chaos to reach a complete thermodynamic equilibrium for a long time. This energy source must have a power that would compensate, in particular, for the expenditure of the turbulent energy dissipated into heat through "molecular" viscosity. For this steady-nonequilibrium regime, almost all of the expendable turbulence energy is known to be transferred without noticeable losses through the inertial interval to the dissipative interval (see Landau and Lifshitz, 1987). Then, an internal equilibrium is established in the structure of the subsystem of turbulent chaos[56] at which $DS^{\text{turb}}/Dt \cong 0$ (the entropy of chaos does not change along the path, $\mathbf{J}_{(S^{\text{turb}})} \equiv \mathbf{J}_b^{\text{turb}}/T^{\text{turb}} \cong \text{const}$ (Kolesnichenko, 2003). This implies that the production $\sigma^i_{S^{\text{turb}}}$ of turbulization entropy (due to internal dissipative processes) is offset by its outflow $\sigma^e_{S^{\text{turb}}}$; i.e., there is no total production of entropy $S^{\text{turb}}$, $\sigma_{S^{\text{turb}}} = \sigma^e_{S^{\text{turb}}} + \sigma^i_{S^{\text{turb}}} \cong 0$. Thus, the subsystem of turbulent chaos exports its entropy into the "external medium," i.e., gives it up to the subsystem of averaged motion. It is important to keep in mind that the conditions of this kind are sufficient for the formation of dissipative coherent structures in an "open" vortex continuum (see Prigogine and Stengers, 1994).

**Deriving the correction function to the coefficient $\boldsymbol{\nu}^{\text{turb}}$.** Thus, for the locally steady state of a developed turbulized flow in the disk, Eq. (166) can be written as

$$\nu^{\text{turb}}(1 - R_f - K_f)\mathring{\mathbf{D}} : \mathring{\mathbf{D}} + \sigma_{\text{rel}} - \varepsilon \approx 0, \tag{167}$$

where, by analogy with the dimensionless dynamic Richardson number

$$R_f \equiv -\frac{g_z\left(\mathbf{q}^{\text{turb}} - \sum_\alpha \langle h_\alpha\rangle \mathbf{J}_\alpha^{\text{turb}}\right)_z}{\langle c_p\rangle\langle T\rangle 2\bar{\rho}\nu^{\text{turb}}\mathring{\mathbf{D}} : \mathring{\mathbf{D}}} = \frac{1}{\text{Sc}^{\text{turb}}}\frac{\frac{g_z}{\langle T\rangle}\left(\nabla_z\langle T\rangle + \frac{g_z}{\langle c_p\rangle}\right)}{2\mathring{\mathbf{D}} : \mathring{\mathbf{D}}} = \frac{\text{Ri}}{\text{Sc}^{\text{turb}}} \tag{168}$$

that allows for the effect of thermal convection on the generation of turbulence in the disk compared to the dynamical factors (here, Ri is the gradient Richardson number), we introduced the dimensionless dynamic Kolmogorov number[57]

$$\text{K}_f \equiv \frac{(g_z/\bar{\rho}_g)(\mathbf{J}_d^{\text{turb}})_z}{2\bar{\rho}\nu^{\text{turb}}\mathring{\mathbf{D}} : \mathring{\mathbf{D}}} = -\frac{1}{\text{Sc}^{\text{turb}}}\frac{(g_z/\bar{\rho}_g)\nabla_z\langle C_d\rangle}{2\mathring{\mathbf{D}} : \mathring{\mathbf{D}}} = -\frac{1}{\text{Sc}^{\text{turb}}}\frac{\langle\sigma\rangle g_z[\nabla_z\bar{s} - \bar{s}\nabla_z\ln\bar{\rho}_g]}{2\mathring{\mathbf{D}} : \mathring{\mathbf{D}}} = \frac{\text{K}}{\text{Sc}^{\text{turb}}}, \tag{169}$$

which is a criterion for the dynamical activity of dust particles in a turbulized shear flow of disk material (Kolesnichenko, 2000). The introduced dimensionless parameters are criteria for the dynamical activity of baroclinic disk material. It follows from (168) that $\text{Ri} < 0$ for $-\nabla_z\langle T\rangle > g_z/\langle c_p\rangle$ (i.e., for unstable thermal stratification of the disk material) and $\text{Ri} > 0$ for $-\nabla_z\langle T\rangle < g_z/\langle c_p\rangle$ (for stable stratification); for indifferent stratification, $\text{Ri} = 0$. However, the presence of suspended fine particles in the flow always causes a decrease in turbulent energy, since the gradient Kolmogorov number is always positive, $\text{K} > 0$ (see Barenblatt and Golitsyn, 1974). Thus, the dimensionless Kolmogorov number K allows for the inverse effect of the stratification (in disk thickness) of the volume concentration of small dust particles on the growth of turbulence in the disk.

Assuming, according to the hypotheses by Kolmogorov (1942), that the kinematic coefficient of turbulent viscosity $\nu^{\text{turb}}$ and the dissipation rate of turbulent energy into heat $\varepsilon$ depend only on two flow parameters, the turbulence energy $b$ and the local turbulence scale length $l(\mathbf{x})$, we obtain (see Eq. (97))

$$\nu^{\text{turb}} = l\sqrt{b}, \quad \varepsilon = \frac{1}{\alpha^2}\frac{b^{3/2}}{l}. \tag{170}$$

Here, since the scale length $l(\mathbf{x})$ is uncertain, the constant in the expression for $\nu^{\text{turb}}$ is taken to be unity and the numerical factor $1/\alpha^2$ is assumed, to a first approximation, to be constant. Let us represent the term (see Eq. (151))

$$\bar{\rho}\sigma_{\text{rel}} \equiv -\overline{\mathbf{\Pi}'_{\text{rel}} : \nabla\mathbf{u}'} = \overline{s\rho_d C_g(\mathbf{ww})' : \nabla\mathbf{u}'} \cong \rho_d\overline{\mathbf{w}}\overline{\mathbf{w}} : (\bar{s}\overline{C'_g\nabla\mathbf{u}'} + \langle C_g\rangle\overline{s'\nabla\mathbf{u}'}),$$

[56]In fact, all of the existing semiempirical theories of turbulence assume (explicitly or implicitly) the existence of an internal equilibrium in the turbulence structure, when the production of turbulence energy is equal to its dissipation at each point.

[57]In fact, only the first term in Eq. (169) is the Kolmogorov number (at $\rho_g = \text{const}$, the number K expresses the relative expenditure of turbulent energy on the suspension of particles by the carrier flow); the second term, which describes the influence of gas inhomogeneity on turbulence, is the so-called Prandtl criterion.

which is responsible for the additional generation of turbulence energy at large Reynolds numbers (in the wakes behind the moving large particles) as[58]

$$\sigma_{rel} = \beta\bar{s}\frac{|\bar{\mathbf{w}}|^2\sqrt{b}}{l}, \quad (171)$$

where $\beta$ is an empirical constant. It should be noted that Eq. (171) is similar in form to the expression[59] that was derived by Varaksin (2003) using a self-similar solution for the far axisymmetric turbulent wake (Yarin and Hetsroni, 1994) and by Derevich (1994) using a semiempirical approach and that is valid only at a very low volume concentration $\bar{s}$ of the disperse phase, when there is no interference of the wakes behind individual particles.

Substituting (170) and (171) in (167) yields

$$2l\sqrt{b}(1 - R_f - K_f)\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} + \beta\bar{s}\frac{|\bar{\mathbf{w}}|^2\sqrt{b}}{l} - \frac{1}{\alpha^2}\frac{b^{3/2}}{l} = 0. \quad (172)$$

Equation (172) breaks up into two equations: $b = 0$ that corresponds to a laminar flow in the disk system, and

$$b = \alpha^2 l^2\left(1 - \alpha^2\beta\bar{s}\frac{|\bar{\mathbf{w}}|^2}{b}\right)^{-1}\left(1 - \frac{\mathrm{Ri} + \mathrm{K}}{\mathrm{Sc}^{\mathrm{turb}}}\right)(2\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}}) \quad (173)$$

that describes the steady turbulent flow of a gas suspension. Eq. (173) has real solutions only at $\mathrm{Ri} + \mathrm{K} < (\mathrm{Ri} + \mathrm{K})_{cr} = \mathrm{Sc}^{\mathrm{turb}}$ (at $\mathrm{Ri} + \mathrm{K} \geq \mathrm{Sc}^{\mathrm{turb}}$, only one real solution, $b = 0$, pertaining to a laminar flow exists). Let the regime be turbulent; then,

$$\sqrt{b} = \alpha l\frac{\sqrt{1 - (\mathrm{Ri} + \mathrm{K})/\mathrm{Sc}^{\mathrm{turb}}}}{\sqrt{1 - \alpha^2\beta^*(d_d^2/l^2)\bar{s}}}\sqrt{2\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}}}. \quad (174)$$

We see from (173) that the presence of large solid particles in the flow always causes an increase in turbulent energy, since the averaged kinetic energy of the relative phase motion ($\propto |\bar{\mathbf{w}}|^2$) for them is comparable to the turbulence energy $b$.

Thus, we obtain for the local coefficient of turbulent viscosity in a gas–dust disk

$$\nu^{\mathrm{turb}}(\mathbf{x}) = \alpha l^2(\mathbf{x})\varphi\varphi_1\sqrt{2\overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}}}, \quad (175)$$

where

$$\varphi = \varphi(\mathrm{Ri}, \mathrm{K}) \equiv \sqrt{1 - (\mathrm{Ri} + \mathrm{K})/\mathrm{Sc}^{\mathrm{turb}}}, \quad (176)$$

$$\varphi_1 = \varphi_1(\bar{s}, C_D, \delta/\xi_d, d_d/l) \equiv \sqrt{1 - \alpha^2 a(C_D\delta/\xi_d)^{4/3}(d_d/l)^2\bar{s}} \quad (177)$$

are the correction dimensionless functions that allow, respectively, for the inverse effect of the dust and heat transfer on the growth of turbulence in the protoplanetary disk ($\varphi$) and for the influence of medium and coarse particles on the turbulent heat and mass transfer in the flow and their contribution to the coefficient of turbulent viscosity for the gas suspension ($\varphi_1$).

## STEADY MOTIONS IN A TURBULIZED SUBDISK

As an illustration of the approach developed here, let us now use the general relations derived above to model the protoplanetary gas–dust cloud that rotated around the proto-Sun at an early stage of its evolution, until it lost the gas component, based on a schematized description of a steady axisymmetric turbulized flow of disk material, which, however, leads to manageable and numerically solvable equations. Since we are eventually interested in the spatial distribution of thermohydrodynamic parameters inside the dust layer (in the subdisk) formed when solid particles settled to the equatorial plane of the proto-Sun in the presence of developed turbulence in the disk, for completeness, it will also be important to consider simple mechanical properties of the rotating gas–dust cloud as a whole. We will analyze the disk system under the following assumptions:

(1) we investigate a slowly evolving gas–dust cloud that rotates around a fixed (in space) $z$ axis with an angular velocity $\Omega(r, z)$;

(2) the rotation is assumed to be so slow that the meridional circulation of the protoplanetary cloud material may be ignored (in essence, the velocity has only the $\varphi$ component for Keplerian accretion disks, i.e., $u_z \ll u_r \ll u_\varphi$;

(3) the magnetic fields play no significant role (the figure of the cloud is known to become flattened in the absence of a macroscopic magnetic field);

(4) the disk configuration is assumed to be stationary in an inertial frame of reference with the origin at the center of mass of the proto-Sun;

(5) the existence of a midplane of symmetry that coincides with the equatorial plane of the proto-Sun defined by the condition $z = 0$ is postulated for a baroclinic disk (the equation of state (136) is valid for its material);

(6) the ratio of the disk half-thickness $h_{\mathrm{disk}}(r)$ to its radius is assumed to be much smaller than unity, $h_{\mathrm{disk}}(r)/r \ll 1$ (the thin-disk condition);

[58]Formula (171) for the term that models the additional generation of turbulence energy by large particles has been suggested for the first time.

[59]Formula (171) is identical to the expression $\sigma_{rel} = \beta^*\bar{s}|\bar{\mathbf{w}}|^3/d_p$ (Varaksin, 2003) at $|\bar{\mathbf{w}}|/\sqrt{b} \equiv d_d/l$. Here, $\beta^* = a(C_D\delta/\xi_d)^{4/3}$; $\xi_d$ is the mixing length for the dust particle concentration; $\delta$ is the wake half-width; $C_D(\mathrm{Re}_d)$ is the coefficient of resistance for the particle ($a = 0.0027$, $\delta/\xi_d = 5$).

(7) we neglect the self-gravity of the disk material compared to the gravitational field of the proto-Sun;

(8) the radiation pressure in the disk is assumed to be much lower than the gas pressure, $p_R \ll p_g$;

(9) the gas–dust disk is optically thick to radiation at all frequencies;

(10) we disregard the chemical reactions and phase transitions and assume the composition of the disk gas phase to be homogeneous;

(11) the large-scale velocity shear of the material related to its differential rotation around the proto-Sun is assumed to be responsible for the turbulization of a Keplerian protoplanetary disk.

*Axisymmetric Motion in a Gas–Dust Disk*

When rotating around the proto-Sun almost exactly according to Kepler's law, each gas-suspension element in the disk slowly moves radially inward, since the deceleration related to the forces of viscous friction between the adjacent cylindrical layers rotating with different angular velocities leads to a redistribution of specific angular momentum and the appearance of a radial mass flow. Thus, the main mass of the disk material slowly (compared to the orbital motion) drifts toward the center of mass along a flat spiral trajectory as the angular momentum, together with the smaller mass, is transferred outward (in view of the conservation law), from the inner disk regions to the outer ones. Concurrently, the turbulent stresses arising from the relative shear of the separate layers of disk material during their orbital motion lead to viscous heat dissipation. The thin-disk condition is known to imply that the disk temperature is relatively low and the pressure gradient is much smaller than the two main mechanical forces, the gravitational and centrifugal ones (see, e.g., Shapiro and Teukolsky, 1983). However, low temperatures are maintained only if the viscous heat dissipated in the turbulized system is effectively radiated outward and is not accumulated in the disk. In the steady state, the bulk of this heat must be radiated by the upper and lower disk surfaces (because the disk is thin and the radiation is directed mainly vertically rather than radially). Thus, a thin accretion disk must be highly nonadiabatic.

Below, we will use an inertial cylindrical $(r, \varphi, z)$ coordinate system with the coordinate origin at the center of mass; the $z = 0$ plane is assumed to coincide with the midplane of symmetry of the disk. We will also assume that the averaged motion of the cosmic fluid has only an azimuthal component,

$$\langle u_r \rangle = 0, \quad \langle u_\varphi \rangle = \Omega(r, z) r, \quad \langle u_z \rangle = 0, \qquad (175)$$

and that the true flow velocity of the gas–dust mixture randomly pulsates about its mean value, varying irregularly in the meridional and azimuthal directions. It can be shown that, if the disk material is in a state of quasi-steady rotation in an inertial frame of reference, then it necessarily possesses axial symmetry $(\partial/\partial\varphi = 0)$: $\bar{s} = \bar{s}(r, z)$, $\bar{\rho}_g = \bar{\rho}_g(r, z)$, $\bar{\rho} = \bar{\rho}(r, z)$, $\bar{p} = \bar{p}(r, z)$, $\langle T \rangle = \langle T(r, z) \rangle$, $\Omega = \Omega(r, z)$, etc. (Tassoul, 1979). Note that the mass conservation law (85) in the steady-state case under consideration always holds, since the motions are axisymmetric and there are no meridional flows, $\nabla \cdot \langle \mathbf{u} \rangle = 0$.

**Momentum conservation equations.** If we take into account the matter–radiation interaction inside the disk up to the terms of the lowest order in $|\langle \mathbf{u} \rangle|/c$ (see footnote[24]), then the three components of the averaged equation of motion (107) can be written as[60]

$$\frac{1}{\bar{\rho}} \frac{\partial}{\partial r}(\bar{p} + p^{\text{turb}}) = r\Omega^2(r, z) - \frac{G\mathcal{M}_\odot r}{(r^2 + z^2)^{3/2}} \equiv g_r \cong 0, \qquad (176)$$

$$\frac{1}{\bar{\rho}} \frac{\partial}{\partial z}(\bar{p} + p^{\text{turb}}) = -\frac{G\mathcal{M}_\odot z}{(r^2 + z^2)^{3/2}} \equiv -g_z,$$
$$\text{где} \quad g_z = \frac{G\mathcal{M}_\odot z}{r^3}\left(1 + \frac{z^2}{r^2}\right)^{-3/2} \cong \Omega^2_{\text{K, mid}}(r) z, \qquad (177)$$

$$\bar{\rho} \frac{\partial}{\partial t}[r\Omega(r, z, t)]$$
$$\cong \frac{1}{r}\frac{\partial}{\partial r}\left\{ r\left[ (\mathbf{R} + \bar{\mathbf{\Pi}}_{\text{rad}})_{r\varphi} - \frac{1}{c^2}(\mathbf{q}_{\text{rad}})_r r\Omega(r, z) \right] \right\}$$
$$+ \frac{\partial}{\partial z}\left[ (\mathbf{R} + \bar{\mathbf{\Pi}}_{\text{rad}})_{z\varphi} - \frac{1}{c^2}(\mathbf{q}_{\text{rad}})_z r\Omega(r, z) \right]$$
$$+ \frac{1}{r}\left[ (\mathbf{R} + \bar{\mathbf{\Pi}}_{\text{rad}})_{\varphi r} - \frac{1}{c^2}(\mathbf{q}_{\text{rad}})_r r\Omega(r, z) \right] \cong 0, \qquad (178)$$

where $\mathbf{g} = \{g_r, 0, -g_z\}$ is the effective gravity (per unit mass) corrected for the centrifugal acceleration; $\Omega_K(r, z) \equiv \sqrt{G\mathcal{M}_\odot/(r^2 + z^2)^{3/2}}$ is the Keplerian angular velocity; $\Omega_{\text{K, mid}}(r) \equiv \Omega_K(r, 0) = \sqrt{G\mathcal{M}_\odot/r^3}$ is the Keplerian angular velocity in the midplane of the disk; $\bar{p}(r, z)$ is the thermal pressure of the disk medium related to the density $\bar{\rho}(r, z)$ and temperature $\langle T(r, z) \rangle$ by the averaged equation of state (55), $\bar{p} = \bar{\rho}\langle \mathcal{R} \rangle \langle T \rangle$; $p^{\text{turb}}(r, z) = 2/3\bar{\rho}\, b =$

[60]Although T Tauri stars probably have a high dust concentration throughout the disk thickness (see, e.g., Beckwith *et al.*, 2000), the additional stresses $\bar{\mathbf{\Pi}}_{\text{rel}}$ related to the relative motion of the gas and the coarse dust effectively act only in the region immediately adjacent to the equatorial plane of the disk (small compared to the entire disk) and, therefore, were omitted in Eq. (178) (see the next section).

$1/3\rho\overline{|\mathbf{u}''|^2}$ is the turbulization pressure (see Eq. (110)); $c$ is the speed of light in a vacuum;

$$\begin{aligned}\mathbf{R} + \overline{\mathbf{\Pi}}_{\rm rad} &\equiv -p^{\rm turb}\mathbf{I} + 2(\bar{\rho}\nu^{\rm turb} + \mu_{\rm rad})\overset{\circ}{\mathbf{D}} \\ &= -\mathbf{i}_r\mathbf{i}_r p^{\rm turb} - \mathbf{i}_\varphi\mathbf{i}_\varphi p^{\rm turb} - \mathbf{i}_z\mathbf{i}_z p^{\rm turb} \\ &+ (\bar{\rho}\nu^{\rm turb} + \mu_{\rm rad})\left\{\mathbf{i}_r\mathbf{i}_\varphi r\frac{\partial\Omega(r,z)}{\partial r} + \mathbf{i}_\varphi\mathbf{i}_r r\frac{\partial\Omega(r,z)}{\partial r}\right. \\ &\left. + \mathbf{i}_\varphi\mathbf{i}_z r\frac{\partial\Omega(r,z)}{\partial z} + \mathbf{i}_z\mathbf{i}_\varphi r\frac{\partial\Omega(r,z)}{\partial z}\right\}\end{aligned} \quad (179)$$

(see Eqs. (110) and (B.8)). In writing Eq. (176), we took into account the fact that the gravitational force is balanced by the centrifugal force in the radial direction (perpendicular to the rotation axis); i.e., the total gas-suspension pressure gradient $\partial(\bar{p} + p^{\rm turb})/\partial r$ is very small and the rotation is almost Keplerian (however, this gradient eventually serves as the driving force of the radial drift of dust particles toward the disk center). On the other hand, since there is no net motion of the gas suspension in the vertical direction (perpendicular to the midplane of the disk), the momentum conservation along the $\mathbf{i}_z$ axis is reduced to the hydrostatic equilibrium condition under which the equilibrium in the $z$ direction is maintained by the pressure gradient.[61] The gradient $\partial(\bar{p} + p^{\rm turb})/\partial z$ determines the main direction of the buoyant force in the gravitational field of the central mass (see Eq. (164)), which contributes, in particular, to the additional generation of turbulent energy through convective instability in the vertical direction (between the midplane and the surfaces of the disk). Thus, it follows from Eqs. (176) and (177) that viscous dissipation does not affect the $r$ and $z$ components of the equation of motion for the entire disk, which break up into separate equations for the averaged radial and vertical motions.

In contrast, the $\varphi$ component of the equation of motion (178) (in essence, the equation for determining the angular velocity $\Omega(r, z)$ for given boundary conditions), which using the transformations (see Appendix B)

$$\begin{aligned}&[\nabla\cdot(\mathbf{R} + \overline{\mathbf{\Pi}}_{\rm rad})]_\varphi \\ &\equiv\left\{\frac{1}{r}\frac{\partial}{\partial r}[r(\mathbf{R} + \overline{\mathbf{\Pi}}_{\rm rad})_{r\varphi}]\right. \\ &\left. + \frac{\partial}{\partial z}[(\mathbf{R} + \overline{\mathbf{\Pi}}_{\rm rad})_{z\varphi}] + \frac{(\mathbf{R} + \overline{\mathbf{\Pi}}_{\rm rad})_{\varphi r}}{r}\right\}\end{aligned} \quad (180)$$

$$\begin{aligned}&= \frac{1}{r}\nabla\cdot[\mu(r,z)r^2\nabla\Omega(r,z,t)], \\ &-\frac{1}{c^2}\{\nabla\cdot[r\Omega(r,z,t)\mathbf{q}_{\rm rad}]\}_\varphi \\ &= -\frac{1}{c^2 r}\nabla\cdot[r^2\Omega(r,z,t)\mathbf{q}_{\rm rad}]\end{aligned} \quad (181)$$

(in what follows, the total coefficient of shear viscosity is denoted by $\mu(r, z) \equiv \bar{\rho}\,\nu^{\rm turb}(r, z) + \mu_{\rm rad}(r, z)$ for short) can be reduced to

$$\begin{aligned}\bar{\rho}\frac{\partial J(r,z,t)}{\partial t} &= \nabla\cdot[\mu(r,z)r^2\nabla\Omega(r,z,t)] \\ &- c^{-2}\nabla\cdot[r^2\Omega(r,z,t)\mathbf{q}_{\rm rad}] \cong 0\end{aligned} \quad (182)$$

(where $\nabla\cdot\mathbf{A} \equiv r^{-1}\partial(rA_r)/\partial r + \partial A_z/\partial z$ is the divergence in cylindrical coordinates), describes the irreversible change in specific angular momentum $J(r, z, t) \equiv [r^2\Omega(r, z, t)]$ through viscous friction and, in addition, allows for the transfer of angular momentum by the total radiative flux $\mathbf{q}_{\rm rad}$ (the second term on the right-hand side of Eq. (179) allows for the loss of angular momentum via the radiation emitted by rotating disk regions). The deceleration by radiation is known (see, e.g., Tassoul, 1982) to be stronger than the angular velocity diffusion through viscosity if $\langle T\rangle/|\nabla\langle T\rangle|$ is small compared to $\langle\Omega\rangle/|\nabla\langle\Omega\rangle|$, which is probably the case only in the disk regions adjacent to its emitting surface $\Sigma$.

Below, however, we will restrict our analysis to steady or quasi-steady motions inside the disk, where, to terms of the order of $|\langle\mathbf{u}\rangle|/c$, Eq. (182) can be reduced to

$$\nabla\cdot[\mu(r,z)r^2\nabla\Omega(r,z)] = 0. \quad (183)$$

Since the shear stress vector must become zero at the outer disk boundary $\Sigma$, the following condition must be satisfied

$$\mathbf{n}\cdot\nabla\Omega(r,z) = 0 \text{ on } \Sigma, \quad (184)$$

where $\mathbf{n}$ is the outer normal to the disk surface.

Let us formulate one curious result pertaining to the problem of a slowly evolving viscous disk in connection with Eqs. (183) and (184). For this purpose, we

[61]The following expression for the half-thickness of a turbulized disk can be derived by substituting the differentials in Eq. (177) with finite differences (i.e., by substituting $\approx \bar{p} + p^{\rm turb}$ for $\Delta(\bar{p} + p^{\rm turb})$, where $(\bar{p} + p^{\rm turb})$ is the total pressure calculated at $z = 0$, and substituting $\Delta z \approx h_{\rm disk}$): $h_{\rm disk} = \sqrt{((\bar{p} + p^{\rm turb})/\bar{\rho})_{z=0}}/\Omega_{\rm K,\,mid} \cong c_{\rm s}\sqrt{1 + 2/3bc_{\rm s}^{-2}}/\Omega_{\rm K,\,mid}$ (cf. Eq. (160)). In this relation, $\Omega_{\rm K,\,mid} = \sqrt{G\mathcal{M}_\odot/r^3}$ is the Keplerian angular velocity; $c_{\rm s}$ is the speed of sound of the gas suspension (see Eq. (57)) in the midplane of the disk.

will write the total dissipative function of the disk in the steady state as

$$D(\Omega) \equiv \int_{\wp} \Phi_{\langle \mathbf{u} \rangle} d\mathbf{x} \equiv \int_{\wp} 2\mu \overset{\circ}{\mathbf{D}} : \overset{\circ}{\mathbf{D}} d\mathbf{x}$$

$$= \int_{\wp} 4\mu(r, z)(D_{r\varphi}^2 + D_{z\varphi}^2) d\mathbf{x} \tag{185}$$

$$= \int_{\wp} \mu(r, z) r^2 \left\{ \left[ \frac{\partial}{\partial r} \Omega(r, z) \right]^2 + \left[ \frac{\partial}{\partial z} \Omega(r, z) \right]^2 \right\} d\mathbf{x},$$

where $\wp$ is the disk volume (see Eqs. (B8) and (B.11)). If we now consider an arbitrary disk rotation law, $\Omega(r, z) + \delta\Omega(r, z, t)$, on which we impose only the constraint that the configuration surface $\Sigma$ and volume $\wp$ are conserved, then the classical results(which, in essence, was obtained by Helmholtz) is that each solution of Eqs. (183) and (184) has the property that the total power (185) dissipated by friction is an absolute minimum compared to the power for any other motion that agrees with the boundary $\Sigma$ and the volume $\wp$.

**The conservation of energy.** To model the internal thermal structure of the protoplanetary disk around the young Sun at the T Tauri stage, we must invoke the energy equation (154), in which the dissipation of turbulent energy is the main internal source of heating. If we disregard the chemical reactions and the evaporation and condensation of disk material, then this equation for quasi-steady axisymmetric motion takes the form[62]

$$\frac{1}{r}\frac{\partial}{\partial r}[r(q_r^{\text{turb}} + q_{\text{rad}, r})] + \frac{\partial}{\partial z}(q_z^{\text{turb}} + q_{\text{rad}, z})$$

$$= \mu(r, z) r^2 \left\{ \left[ \frac{\partial}{\partial r} \Omega(r, z) \right]^2 + \left[ \frac{\partial}{\partial z} \Omega(r, z) \right]^2 \right\} + Q_{\odot}$$

or

$$\frac{\partial}{\partial z}(q_z^{\text{turb}} + q_{\text{rad}, z}) = \mu(r, z) r^2 \times \left\{ \left[ \frac{\partial}{\partial r} \Omega(r, z) \right]^2 + \left[ \frac{\partial}{\partial z} \Omega(r, z) \right]^2 \right\} + Q_{\odot}, \tag{186}$$

since the radiation for a thin disk is directed mainly vertically rather than radially (see also Eq. (198)).

In Eqs. (185) and (186), the possible local heating sources of the gas–dust cloud, which are related, in particular, to the absorption of solar electromagnetic and corpuscular radiation by the gas–dust disk components and its subsequent transformation through various radiative processes, reradiation, scattering, photochemical and chemical reactions, etc., are denoted by $Q_{\odot}$. The complexity and multiplicity of chemical and photochemical reactions in the protoplanetary disk medium is generally attributable to the presence of basic chemical elements of the Solar system that constituted the initial gas-mixture components and to the existence of ionization (dissociation) agents in the form of energetic photons and photoelectrons (photolysis products) (see, e.g., Willacy *et al.*, 1998). Their absorption leads to the dissociation, ionization, and/or excitation of rotational and vibrational levels of the gas-mixture components; each of these reactions can proceed in both forward and reverse directions. In practical calculations, by no means all of the elementary processes responsible for the thermal balance of a disk medium can be adequately taken into account in the corresponding models. For this reason, when physically self-consistent problems of modeling the evolution of the chemical composition and hydrodynamics of a disk are formulated, one of the most important problems is to accurately take into account the contributions from the matter–radiation interaction in the structure of the energy equation to determine the so-called heating function of the material that allows for the fraction of the absorbed solar radiation transformed into heat (see, e.g., Marov and Kolesnichenko, 1987). Estimating this function involve well-known difficulties and requires concretizing the chemical stage of the disk evolution.

The turbulent heat flux $q_z^{\text{turb}}$ and the radiative energy flux $q_{\text{rad}, z}$ emitted by the disk are defined by Eqs. (127) and (128)

$$\begin{cases} q_z^{\text{turb}}(r, z) = -\bar{\rho}\langle c_p \rangle \dfrac{\nu^{\text{turb}}(r, z)}{\text{Sc}^{\text{turb}}} \\ \times \left( \dfrac{\partial}{\partial z}\langle T \rangle + \dfrac{\Omega_{\text{K, mid}}^2(r)}{\langle c_p \rangle} z \right) \\ q_{\text{rad}, z}(r, z) = -\chi_{\text{rad}}(r, z) \dfrac{\partial}{\partial z}\langle T \rangle, \end{cases} \tag{187}$$

where $\chi_{\text{rad}}(r, z) = 4ca\langle T \rangle^3 / 3\tilde{\kappa}\bar{\rho}$ is the coefficient of radiative heat conductivity for the disk medium; $\tilde{\kappa}$ is the total Rosseland mean opacity of the gas suspension (see Eq. (72)), which depends significantly on the presence and height distribution of dust particles in the protoplanetary cloud (see, e.g., Pollack *et al.*, 1985); $\mu_{\text{rad}}(r, z) = 4a\langle T \rangle^4 / 15c\tilde{\kappa}\bar{\rho}$ is the coefficient of radiative viscosity. Equation (187) should be complemented by the following boundary conditions in the midplane (in

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[62]The possible additional disk heating source related to the term $\bar{\mathbf{\Pi}}_{\text{rol}} : \mathbf{D}$ effectively acts only in the dust subdisk (i.e., in a volume small compared to the entire disk) and, therefore, was omitted in Eq. (186).

view of the disk symmetry) and on the upper surface of the disk:

$$q_z^{\text{turb}}\big|_{z=0} = \frac{\partial}{\partial z}\langle T\rangle = 0, \;^{(*)}$$
$$\mathbf{n}\cdot(\mathbf{q}^{\text{turb}} + \mathbf{q}_{\text{rad}}) = \sigma\langle T\rangle^4 - f_0\frac{L_\odot}{4\pi r^2}, \text{ on } \Sigma \;^{(**)} \tag{188}$$

where $\sigma$ is the Stefan–Boltzmann constant; $L_\odot$ is the solar luminosity), which allow for the heat balance at the boundaries. The first term on the right-hand side of Eq. $^{(**)}$ allows for the blackbody radiation of the disk surface, while the second term describes the attenuated radiative flux from the protostar incident on the disk surface; the attenuation factor (Kusaka *et al.*, 1970)

$$f_0 = \left[\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{h_{\text{disk}}(r,z)}{r} + \frac{2R_\odot^{\text{proto}}}{3\pi r}\right)\right]\frac{L_\odot^{\text{proto}}}{L_\odot} \tag{189}$$

depends on the disk geometry and the radius $R_\odot^{\text{proto}}$ and luminosity $L_\odot^{\text{proto}}$ of the proto-Sun (in particular, for $L_\odot^{\text{proto}} = 7L_\odot$ and $R_\odot^{\text{proto}} = 5R_\odot$ (see Watanabe *et al.*, 1990), $f_0 = 0.1$ at $r = 1$ AU).

**The dust transfer equation.** The averaged transfer equation (92*) for the dust concentration $\langle C_{\text{d}}\rangle = \rho_{\text{d}}\bar{s}/\bar{\rho}$ of the disk material should be used to model the evolution of a turbulized gas–dust cloud, particularly during the formation of a dust subdisk (of thickness $2h_{\text{subdisk}}$, where $h_{\text{subdisk}}$ is the upper boundary of the dust subdisk, $h_{\text{disk}} > h_{\text{subdisk}}$) near its midplane. If we disregard the evaporation and condensation ($\bar{\sigma}_{\text{dg}} = 0$) of solid particles, then there is a balance between the dust settling $\bar{\mathbf{J}}_{\text{d}} = \overline{\tilde{\rho}_{\text{d}}\mathbf{w}_{\text{d}}} \cong \bar{\rho}_{\text{g}}\langle C_{\text{d}}\rangle\bar{\mathbf{w}}$ and the turbulent mixing $\mathbf{J}_{\text{d}}^{\text{turb}} = -\bar{\rho}D_{\text{d}}^{\text{turb}}\nabla\langle C_{\text{d}}\rangle$ in the steady state (see Eq. (86)); Eq. (92*) then takes the form

$$\nabla\cdot\left(\bar{\rho}_{\text{g}}\langle C_{\text{d}}\rangle\bar{\mathbf{w}} - \frac{\bar{\rho}\nu^{\text{turb}}}{\text{Sc}^{\text{turb}}}\nabla\langle C_{\text{d}}\rangle\right) = 0 \tag{190}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\bar{s}\langle C_{\text{g}}\rangle\bar{w}_r - \frac{\bar{\rho}\nu^{\text{turb}}}{\text{Sc}^{\text{turb}}}\frac{\partial}{\partial r}\left(\frac{\bar{s}}{\bar{\rho}}\right)\right)\right] + \frac{\partial}{\partial z}\left(\bar{s}\langle C_{\text{g}}\rangle\bar{w}_z - \frac{\bar{\rho}\nu^{\text{turb}}}{\text{Sc}^{\text{turb}}}\frac{\partial}{\partial z}\left(\frac{\bar{s}}{\bar{\rho}}\right)\right) = 0, \tag{191}$$

where $\bar{\mathbf{w}} \equiv (\bar{\mathbf{u}}_{\text{d}} - \bar{\mathbf{u}}_{\text{g}})$ is the averaged relative velocity of the dust and gas defined by Eq. (101); in cylindrical coordinates, it takes the form

$$\bar{\rho}\theta_{\text{gd}}\bar{w}_r(r,z) \cong \bar{w}_\varphi(r,z)\Omega(r,z) + \frac{1}{\bar{\rho}_{\text{g}}}\frac{\partial\bar{p}}{\partial r} \cong \bar{w}_\varphi(r,z)\Omega_{\text{K, mid}}(r) - \frac{\bar{\rho}}{\bar{\rho}_{\text{g}}}r\eta\Omega_{\text{K, mid}}^2(r), \tag{192}$$

$$\bar{\rho}\theta_{\text{gd}}\bar{w}_\varphi(r,z) = -\frac{1}{2}\bar{w}_r(r,z)\Omega(r,z) \cong -\frac{1}{2}\bar{w}_r(r,z)\Omega_{\text{K, mid}}(r), \tag{193}$$

$$\bar{\rho}\theta_{\text{gd}}\bar{w}_z(r,z) = \frac{1}{\bar{\rho}_{\text{g}}}\frac{\partial\bar{p}}{\partial z} \cong -\frac{\bar{\rho}}{\bar{\rho}_{\text{g}}}\Omega_{\text{K, mid}}^2(r)z. \tag{194}$$

The $r$, $\varphi$, and $z$ components of the averaged relative velocity $\bar{\mathbf{w}}$ derived using the equations of motion (176) and (177) then appear as

$$\begin{cases} \bar{w}_r(r,z) \cong -\dfrac{\bar{\rho}}{\bar{\rho}_{\text{g}}}\eta\dfrac{\zeta}{1+\zeta^2}r\Omega_{\text{K, mid}}(r) \\ \bar{w}_\varphi(r,z) \cong \dfrac{\bar{\rho}}{2\bar{\rho}_{\text{g}}}\eta\dfrac{\zeta^2}{1+\zeta^2}r\Omega_{\text{K, mid}} \\ \bar{w}_z(r,z) \cong -\zeta z\Omega_{\text{K, mid}}(r), \end{cases} \tag{195}$$

where the parameter $\zeta = \Omega_{\text{K, mid}}/\bar{\rho}\,\theta_{\text{gd}} \ll 1$, because the time it takes for quasi-equilibrium motion of the dust and gas to be established ($1/\bar{\rho}\,\theta_{\text{gd}}$) in the disk is much shorter than the Keplerian period ($2\pi/\Omega_{\text{K, mid}}$), which determines the slow variation time scales of the macroscopic flow parameters. Here, we used the following approximate equality that follows from Eq. (176):

$$\Omega(r,z) = \Omega_{\text{K, mid}}(r)[1-\eta]^{1/2} \cong \Omega_{\text{K, mid}}(r). \tag{196}$$

In this equality, the small parameter is defined as[63]

$$\eta \equiv -(r\Omega_{\text{K, mid}}^2\bar{\rho})^{-1}\partial\bar{p}/\partial r = -\bar{\gamma}\left(\frac{\mathcal{H}}{r}\right)^2\left(f + q + \frac{q+3}{2}\frac{z^2}{\mathcal{H}^2}\right) = 3.62\times10^{-3}r_{\text{AU}}^{1/2}, \tag{197}$$

where the second estimation representation was obtained using Eq. (210) (cf. Nakagawa *et al.*, 1986; Takeuchi and Lin, 2002).

The diffusion equation (190) can be simplified depending on whether the gas or dust component dominates in the disk region under consideration.

[63] It can be shown that including the pressure of turbulent chaos, $p^{\text{turb}} = 2/3\,\bar{\rho}\,b$, will not change Eq. (194) in the case of developed turbulence.

**A formula for the coefficient of turbulent viscosity in the disk.** The coefficient of turbulent viscosity in Eqs. (179), (182), (187), and (190) is defined by relation (175). In the axisymmetric case under consideration, the latter takes the form

$$\nu^{\text{turb}}(r, z) = \alpha l^{*2} r \sqrt{\left\{\left[\frac{\partial}{\partial r}\Omega(r, z)\right]^2 + \left[\frac{\partial}{\partial z}\Omega(r, z)\right]^2\right\}}, \quad (198)$$

where

$$l^*(z) \equiv l(z)[1 - (\text{Ri} + \text{K})/\text{Sc}^{\text{turb}}]^{0.25}, \quad (199)$$

$$\text{Ri} \equiv \frac{\Omega^2_{\text{K, mid}} z}{r^2} \frac{1}{\langle T\rangle} \frac{\dfrac{\partial\langle T\rangle}{\partial z} + G_{\text{a}}}{\left[\dfrac{\partial}{\partial r}\Omega(r, z)\right]^2 + \left[\dfrac{\partial}{\partial z}\Omega(r, z)\right]^2}, \quad (201)$$

$$\text{K} \equiv -\langle\sigma\rangle \frac{\Omega^2_{\text{K, mid}} z}{r^2} \frac{\dfrac{\partial\bar{s}}{\partial z} - \bar{s}\dfrac{\partial}{\partial z}\ln\bar{\rho}_{\text{g}}}{\left[\dfrac{\partial}{\partial r}\Omega(r, z)\right]^2 + \left[\dfrac{\partial}{\partial z}\Omega(r, z)\right]^2}, \quad (202)$$

$$G_{\text{a}} \equiv \frac{g_z}{\langle c_p\rangle} = -\frac{1}{\langle c_p\rangle}\frac{G\mathcal{M}_\odot z}{r^3}\left(1 + \frac{z^2}{r^2}\right)^{-3/2} \cong \frac{1-\bar{\gamma}}{\bar{\gamma}}\frac{1}{\langle\mathcal{R}\rangle}\Omega^2_{\text{K, mid}} z \quad (203)$$

is the adiabatic temperature gradient in the protoplanetary gas–dust disk; $\bar{\gamma} = \langle c_p\rangle/(\langle c_p\rangle - \langle\mathcal{R}\rangle)$ and $\langle\mathcal{R}\rangle = \mathcal{R}_{\text{g}}\bar{\rho}_{\text{g}}/\bar{\rho}$ are, respectively, the adiabatic index and the "gas constant" for the averaged two-phase continuum. We see from Eqs. (198)–(201) that for an adiabatic temperature distribution in height,

$$-\frac{\partial\langle T\rangle}{\partial z} = \left(-\frac{\partial\langle T\rangle}{\partial z}\right)_{\text{ad}} = \frac{\Omega^2_{\text{K, mid}}(r)}{\langle c_p\rangle} z, \quad (204)$$

the Richardson number is Ri = 0 and the temperature gradient in the disk has no effect on the turbulent transfer coefficients. In the case of temperature-unstable stratification (Ri < 0) of the gas–dust disk, where the temperature gradients are superadiabatic,

$$\frac{\partial\langle T\rangle}{\partial z} + \frac{\Omega^2_{\text{K, mid}}(r)}{\langle c_p\rangle} z = f\frac{\partial\langle T\rangle}{\partial z} \quad (205)$$

(the factor $f$, which characterizes the excess of the vertical temperature gradient in the disk above the adiabatic one, can reach $f = 0.2$ at $r \simeq 10$ AU (see Makalkin and Dorofeeva, 1995, 1996)), the turbulence energy increases through the energy of instability in the direction perpendicular to the midplane of the disk (a convective source of turbulence); the coefficient of turbulent viscosity increases simultaneously. At the same time, gas-suspension inhomogeneity always causes the turbulent energy to decrease, since the Kolmogorov number is greater than zero, K > 0. The reciprocal Schmidt number $1/\text{Sc}^{\text{turb}}$ in Eq. (199) may be set equal to unity when the shear stresses in the case of differential Keplerian disk rotation are the main turbulence mechanism; however, it can be a factor of 2 or 3 larger when thermal convection in the vertical direction is responsible for the turbulence (see, e.g., Shakura *et al.*, 1978).

Let us now show that the vertical gradient in angular velocity $\partial\Omega(r, z)/\partial z$ in the dissipative function $\Phi_{\langle\mathbf{u}\rangle} = 2\bar{\rho}\,\nu^{\text{turb}}\mathring{\mathbf{D}} : \mathring{\mathbf{D}}$ of an axisymmetric disk may occasionally be disregarded compared to its radial gradient $\partial\Omega(r, z)/\partial r$ for the following reasons. If we assume (for our estimation) that the disk is isothermal in the vertical direction and neglect the terms of order $(z/r)^2$ or higher, then the following formula (known for a laminar flow) can be derived for the vertical gas-suspension density distribution in a turbulized disk[64] from Eq. (177):

$$\bar{\rho}(r, z) = \bar{\rho}(r, 0)\exp\left\{-\frac{\Omega^2_{\text{K, mid}}(r)}{2\langle\mathcal{R}\rangle(\langle T\rangle + T^{\text{turb}})} z^2\right\} = \bar{\rho}(r, 0)\exp\left\{-\frac{z^2}{2\mathcal{H}^2}\right\}, \quad (206)$$

where

$$\mathcal{H} = \frac{\sqrt{\langle\mathcal{R}\rangle(\langle T\rangle + T^{\text{turb}})}}{\Omega_{\text{K, mid}}(r)} = \frac{\tilde{c}_{\text{s}}}{\bar{\gamma}^{1/2}\Omega_{\text{K, mid}}(r)} \quad (207)$$

is the local scale height for the disk,

$$\tilde{c}_{\text{s}} \equiv \sqrt{(\partial(\bar{p} + p^{\text{turb}})/\partial\bar{\rho})_{\langle S\rangle}} = \sqrt{\bar{\gamma}\langle\mathcal{R}\rangle(\langle T\rangle + T^{\text{turb}})} \quad (208)$$

is the isothermal speed of sound in the turbulized medium. The density, pressure, temperature, opacity, etc. in any accretion disk have different spatial distributions, depending on its nature and the distance from protostar and/or from the midplane of the disk. We will assume that the radial distributions of such structure parameters follow a power law (this is a common assumption in astrophysical literature (see, e.g., Takeuchi and Lin, 2002)); then,

$$\begin{cases} \tilde{c}^2_{\text{s}}(r) = \tilde{c}^2_{\text{s. AU}} r^q_{\text{AU}}, \quad q = -0.5 \\ \bar{\rho}(r, z) = \bar{\rho}_{\text{AU}} r^f_{\text{AU}} \exp\{-z^2/2\mathcal{H}^2\} \\ \bar{\rho}_{\text{AU}} = 2.83 \times 10^{-10}\ \text{g cm}^{-3}, \quad f = -2.25 \\ \mathcal{H}(r) = \mathcal{H}_{\text{AU}} r^{(q+3)/2}_{\text{AU}}, \quad \mathcal{H}_{\text{AU}} = 3.33 \times 10^{-2}\ \text{AU}, \end{cases} \quad (209)$$

[64] The thermodynamic temperature $T^{\text{turb}}$ (a parameter characterizing the intensity of the noise of the subsystem of turbulent chaos generated by its "thermal structure") is related to the turbulence energy $b$ by $2/3\,\bar{\rho}\,b = p^{\text{turb}} = \bar{\rho}\,\langle\mathcal{R}\rangle T^{\text{turb}}$ (see Kolesnichenko and Marov, 1999).

where $r_{AU}$ is the radius measured in AU. Using (206), Eq. (176) can be rewritten as

$$\Omega^2(r,z) = \Omega^2_{K,\,mid}\left[1 + \frac{1}{\bar{\rho} r \Omega^2_{K,\,mid}}\frac{\partial}{\partial r}(\bar{p} + p^{turb})\right]$$

$$= \Omega^2_{K,\,mid}\left[1 + \frac{\tilde{c}_s^2}{\bar{\rho} r \Omega^2_{K,\,mid}}\frac{\partial \bar{\rho}}{\partial r}\right]$$

$$\cong \Omega^2_{K,\,mid}\left[1 + \frac{\bar{\gamma}\mathcal{H}^2}{r^2}\left(f + q + \frac{q+3}{2}\frac{z^2}{\mathcal{H}^2}\right)\right];$$

whence

$$\Omega(r,z) = \Omega_{K,\,mid}\left[1 + \frac{\bar{\gamma}}{2}\left(\frac{\mathcal{H}}{r}\right)^2\left(f + q + \frac{q+3}{2}\frac{z^2}{\mathcal{H}^2}\right)\right]. \quad (210)$$

It follows from Eq. (210) that

$$\frac{\partial}{\partial z}\Omega(r,z) \sim \frac{\mathcal{H}}{r}\frac{\partial}{\partial r}\Omega(r,z), \quad (\mathcal{H}/r \ll 1), \quad (211)$$

which allows the vertical angular velocity gradient in the dissipative function $\Phi_{\langle \mathbf{u} \rangle}$ to be disregarded.

Thus, the following approximate relation for the coefficient of turbulent viscosity is valid for the bulk of the disk (except the regions close to the proto-Sun):

$$\nu^{turb}(r,z) = \alpha l^{*2} r\left|\frac{\partial}{\partial r}\Omega(r,z)\right|,$$
$$l^*(z) \equiv l(z)[1 - (\mathrm{Ri} + \mathrm{K})/\mathrm{Sc}^{turb}]^{0.25}. \quad (212)$$

For Eq. (212) to be formally identical to the Shakura–Sunyaev formula (160), which is valid for gas-phase accretion disks, we must set K = 0 and Ri = 0 in Eq. (212). If we now substitute the angular velocity of Keplerian rotation $\Omega_{K,\,mid}(r) = (G\mathcal{M}_\odot)^{1/2} r^{-3/2}$ (note that $r|\partial\Omega_{K,\,mid}(r)/\partial r| = -3/2\Omega_{K,\,mid}$) in (212) and use $l = h_{disk} = \sqrt{(\bar{p}/\bar{\rho})|_{z=0}}/\Omega_{K,\,mid}$ as the turbulence scale length (see footnote[61]), then we will obtain $\nu^{turb} = 3/2\alpha(\bar{p}/\bar{\rho})|_{z=0}/\Omega_{K,\,mid}$; then,

$$R_{r\varphi} = \bar{\rho}\nu^T r(\partial\Omega_{K,\,mid}(r)/\partial r) = -9/4\alpha\bar{p}|_{z=0}, \quad (213)$$

which is identical to Eq. (160) (since the dimensionless parameter α cannot be accurately calculated and remains a free parameter in the disk structure equations, the factor 9/4 in Eq. (213) is of no fundamental importance).

So, Eqs. (55), (176), (177), (183), (186), and (190) form a system of six equations with six unknown functions, $\bar{p}(r,z)$, $\bar{\rho}(r,z)$, $\langle T\rangle(r,z)$, $\bar{s}$, $\bar{\rho}_g$, and $\Omega(r,z)$. Thus, in principle, the structure of a gas–dust disk with the dust component is completely defined by these equations together with the boundary conditions and relations (198) and (21*) for the turbulent transfer coefficients $\nu^{turb}$ and the coefficient of resistance $\theta_{dg}$ for a smooth spherical particle. A complete solution of the formulated problem requires using numerical methods and will be presented in a special publication.

*The Regime of Limiting Saturation of a Gas–Dust Disk by Fine Dust Particles*

As a simple example that illustrates the potentialities of the approach developed here, let us qualitatively consider the model problem of the height distribution of suspended fine dust particles in a steady gas–dust flow (for temperature-neutral disk stratification, Ri = 0) in a thin layer of "cosmic fluid" located near a dust–gas subdisk (a layer of enhanced dust concentration, but lower than the critical value at which gravitational instability arises). We will assume that the concentration of solid particles in the subdisk atmosphere and in the subdisk itself is so high that the inverse effect of the disperse phase on the turbulent flow dynamics should be taken into account to describe the gas-suspension motion. Below, for simplicity, we assume the disk material in the subdisk atmosphere to be isothermal and the gas phase to be incompressible. In addition, we will keep in mind that the sedimentation of solid particles takes place without their large radial migration; therefore, the dust diffusion flux in the vertical direction $\partial\overline{\tilde{\rho}_d w_{dz}}/\partial z \gg \partial\overline{\tilde{\rho}_d w_{dr}}/\partial r$. For steady motion of the dust component in the atmosphere, the regular gravitational settling of particles to the subdisk (the value $aSc^{turb}$, where $a = -w_z$ is the gravitational settling velocity, is assumed to be constant below) must then be balanced by their upward turbulent transfer, i.e., $\bar{J}_{dz} + J^{turb}_{dz} = 0$; hence, using Eqs. (96) and (190), we obtain for the relative velocity in the vertical direction

$$-a = \frac{\nu^{turb}}{\mathrm{Sc}^{turb}}\frac{\partial\ln(\bar{s}/\bar{\rho}_g)}{\partial z} \cong \frac{\nu^{turb}}{\mathrm{Sc}^{turb}}\frac{\partial\ln\bar{s}}{\partial z}. \quad (214)$$

Since the additional stresses $\overline{\mathbf{\Pi}}_{rel}$ associated with the relative motion of the gas and coarse dust effectively act in the subdisk, the meridional φ component of the Reynolds equation (178) for the inner subdisk regions is reduced to the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\{r[(\mathbf{R} + \overline{\mathbf{\Pi}}_{rel})_{r\varphi}]\} + \frac{\partial}{\partial z}[(\mathbf{R} + \overline{\mathbf{\Pi}}_{rel})_{z\varphi}]$$
$$+ \frac{1}{r}[(\mathbf{R} + \overline{\mathbf{\Pi}}_{rel})_{\varphi r}] \cong 0, \quad (215)$$

where, in view of relations (195), the corresponding components of the relative stress tensor $\overline{\mathbf{\Pi}}_{rel} \cong -\bar{s}\rho_d\langle C_g\rangle\overline{\mathbf{w}}\overline{\mathbf{w}}$ take the form

$$(\mathbf{\Pi}_{rel})_{z\varphi} = \bar{s}\rho_d\langle C_g\rangle a\bar{w}_\varphi$$
$$= \bar{s}\rho_d\frac{\zeta^3\eta}{2(1+\zeta^2)}zr\Omega^2_{K,\,mid}(r), \quad (216)$$

$$(\mathbf{\Pi}_{\text{rel}})_{\varphi r} = -\bar{s}\rho_d\langle C_g\rangle \overline{w}_r\overline{w}_\varphi = \frac{\bar{s}\rho_d\bar{\rho}}{2\bar{\rho}_g}\frac{\zeta^3\eta^2}{(1+\zeta^2)^2}r^2\Omega^2_{\text{K, mid}}(r). \quad (217)$$

Since the main direction in the entire disk is radial (see Safronov and Guseinov, 1990), we may set for the subdisk

$$\frac{\partial}{\partial z}[(\mathbf{R}+\overline{\mathbf{\Pi}}_{\text{rel}})_{z\varphi}] \cong 0. \quad (218)$$

This equation shows that the flux density of the $\varphi$ momentum component along the vertical axis will be the same at all distances from the $z = 0$ plane to the $z = h_{\text{subdisk}}$ "surface" of the subdisk: $(\mathbf{R} + \overline{\mathbf{\Pi}}_{\text{rel}})_{z\varphi} = (\mathbf{R} + \overline{\mathbf{\Pi}}_{\text{rel}})_{z\varphi}|_{z=0}$ = const. Since near the midplane and at moderately large $z$, in view of Eqs. (162) and (212),

$$R_{z\varphi}|_{z\approx 0} \equiv \bar{\rho}(r,0)\nu^{\text{turb}}(r,0)r\left\{\frac{\partial\Omega(r,z)}{\partial z}\right\}\Bigg|_{z\approx 0} = 0, \quad (219)$$

the Reynolds equation (215) for the subdisk atmosphere can be taken in the form

$$\nu^{\text{turb}}\frac{\partial}{\partial z}[r\partial\Omega(r,z)/\partial z] \equiv V_*^2, \quad (220)$$

where

$$V_*^2 = (\overline{\mathbf{\Pi}}_{\text{rel}})_{z\varphi}|_{z=h_{\text{subdisk}}} \cong \frac{\rho_d\zeta^3\eta}{2\bar{\rho}(r,0)(1+\zeta^2)} \times r\Omega^2_{\text{K, mid}}(r)[z\bar{s}(r,z)]_{z=h_{\text{subdisk}}}, \quad (221)$$

$V_*$ is the so-called dynamic velocity (a natural velocity scale for the flow near the subdisk "surface"). In Eq. (220), we disregarded the contribution from the relative stresses compared to the Reynolds turbulent stresses.

Eliminating the meridional velocity gradient $\partial[r\Omega(r, z)]/\partial z$ from the expression $b = \alpha^2 l^2(1 - K_f)[r\partial\Omega(r, z)/\partial z]^2$ (see (173)) using Eq. (220) and the relation $\nu^{\text{turb}} = l\sqrt{b}$, we obtain

$$b = \alpha V_*^2\sqrt{1-\text{K}_f}, \quad (222)$$

where

$$\text{K}_f \equiv -\frac{\langle\sigma\rangle}{\text{Sc}^{\text{turb}}}\frac{G\mathcal{M}_\odot z}{r^3}\frac{\partial\bar{s}/\partial z}{[r\partial\Omega(r,z)/\partial z]^2} = \frac{G\mathcal{M}_\odot z}{r^4}\frac{\langle\sigma\rangle\bar{s}a}{V_*^2\partial\Omega(r,z)/\partial z} \quad (223)$$

is the dynamic Kolmogorov number. Thus, the $z$ distributions of the disk structure parameters can be calculated using Eqs. (214), (220), (222), and (223), which we will write as

$$\begin{cases} l\sqrt{b}\dfrac{\partial}{\partial z}[r\partial\Omega(r,z)/\partial z] = V_*^2 \\ \dfrac{1}{\text{Sc}^{\text{turb}}}l\sqrt{b}\dfrac{\partial\bar{s}}{\partial z} + a\bar{s} = 0, \quad l(z) = \dfrac{\gamma^*\kappa}{\alpha^{1/2}}z\Phi(\text{K}_f) \\ b = \alpha V_*^2\sqrt{1-\text{K}_f} \\ \text{K}_f = \langle\sigma\rangle\dfrac{G\mathcal{M}_\odot z}{r^4}\dfrac{\bar{s}a}{V_*^2\partial\Omega(r,z)/\partial z}. \end{cases} \quad (224)$$

Here, the local turbulence scale length $l(z)$ is expressed (under the assumption of complete self-similarity in local, Re = $u_0z/\nu$, and global, $\text{Re}_{\text{glob}} = u_0L/\nu$, Reynolds numbers) in terms of a universal function $\Phi(\text{K}_f)$ of the Kolmogorov number $\text{K}_f$; clearly, $\Phi(0) = 1$. Since the turbulence scale length under the effect of the gas-suspension dust component decreases (see Eq. (199)), the function $\Phi(\text{K}_f)$ must also decrease with increasing argument.

Let us now determine the $z$ distribution of the dust volume concentration and the angular velocity of a turbulized flow carrying fine suspended particles for a steady flow in the subdisk atmosphere. To integrate the system of equations (224) generally requires knowledge of the boundary conditions on the subdisk surface. For example, if we eliminate $\nu^{\text{turb}}$ from (214) and (220), then we will obtain

$$\frac{\partial\ln\bar{s}}{\partial z} = -\omega\frac{\gamma^*\kappa}{V_*}\frac{\partial}{\partial z}[r\partial\Omega(r,z)], \quad (225)$$

where

$$\omega \equiv \text{Sc}^{\text{turb}}a/\gamma^*\kappa V_* \quad (226)$$

is a dimensionless parameter. Integrating Eq. (225) yields

$$\bar{s}(r,z) = \bar{s}(r,z)|_{z=h_{\text{subdisk}}} \times \exp\left\{-\frac{\omega\gamma^*\kappa}{V_*}r[\Omega(r,z)-\Omega_{\text{K, mid}}(r,z)|_{z=h_{\text{subdisk}}}]\right\}. \quad (227)$$

At the same time, system (224) has properties peculiar to this type of gas-suspension flow that allow a self-similar solution independent of the boundary conditions to be found.[65] This is because it contains only the angular velocity gradient rather than the velocity itself. This, in turn, implies that for an unlimited store of particles near the dust subdisk, in view of the inverse effect of particles on the flow dynamics, one might expect the existence of a gas-suspension flow regime in it at which

[65]The pattern of turbulent flows containing a solid admixture was first studied by Kolmogorov (1954) when calculating the motion of sediments and by Barenblatt and Golitsyn (1973) for atmospheric problems.

the turbulent flow absorbs the maximum possible amount of dust at a given dynamic velocity and other flow parameters (see Barenblatt and Golitsyn, 1974). This regime, which was called the "regime of limiting saturation" in the literature, must be described by a special solution of system (224) that is defined by the parameters appearing only in the differential equations (224) themselves.

A group analysis of system (224) shows that it has the solution

$$\partial\Omega(r, z)/\partial z = C_1(r)/rz, \quad \bar{s} = C_2(r)/z^2, \tag{228}$$

where $C_1$ and $C_2$ are the $z$-independent functions of $r$ to be determined. Substituting these expressions in (224) yields

$$C_1 = \frac{V_*}{\gamma^*\kappa\Phi(\mathrm{K}_f)(1-\mathrm{K}_f)^{1/4}} = \frac{2V_*}{\gamma^*\kappa\omega}; \tag{229}$$

whence follows the functional equation

$$\Phi(\mathrm{K}_f)(1-\mathrm{K}_f)^{1/4} = \omega/2 \tag{230}$$

intended to determine the specific (for the regime of motion under study) Kolmogorov number $\mathrm{K}_f$ (calculated at a given distance $r$ from the protostar).

Since $\Phi$ is a nonincreasing function of its argument, $\Phi(0) = 1$, and since the number $\mathrm{K}_f$, by its physical meaning, lies between zero and unity, the functional equation at $\omega > 2$ (which corresponds to a low flow velocity or large particles) has no root; however, at $\omega < 2$ (the condition for the existence of the limiting saturation regime), one root exists, $\mathrm{K}_f = \mathrm{K}_f^*$. Given (229), it follows from the first relation (228) (at $\omega < 2$) that

$$\Omega(r, z) = \Omega_{\mathrm{K,\,mid}}(r) + \frac{V_*(r)}{\gamma^*\kappa\omega r}\ln z. \tag{231}$$

Using this relation and Eq. (226), we can easily obtain the following limiting steady-state height distribution of fine dust particles in the thin equatorial layer of the subdisk:

$$\bar{s} = \frac{C_2(r)}{z^2} = \frac{V_*^4 \mathrm{K}^*}{\langle\sigma\rangle a^2 \Omega^2_{\mathrm{K,\,mid}}}\frac{1}{z^2}, \tag{232}$$

which the flow approaches for an unlimited store of dust on the underlying surface.

Formula (231) shows that the $z$ velocity distribution in the "near-surface" subdisk atmosphere in a turbulent flow heavily loaded with particles is logarithmic (as must be the case in a turbulized fluid near a "wall"), but the presence of dust seemingly results in a decrease in the Karman coefficient $\kappa$. This can be interpreted in such a way that the gas-suspension flow under the effect of dust particles for the same external conditions (the same dynamic velocity $V_*$) is accelerated compared to a flow of "pure" gas. In other words, the velocity gradients near the subdisk surface increase, which contributes to the saltation effect, the separation and rise of a large number of fine dust particles into its atmosphere In turn, the presence of such a dust cloud with an enhanced concentration of suspended fine particles contributes to the intensification of all the possible processes of turbulent coagulation, which lead to an increase in the inertia of solid particles and to their effecting settling to the subdisk. Thus, the possible regime of limiting saturation of a rotating gas–dust cloud near the subdisk by small dust particles is an additional mechanism that speeds up the formation of the subdisk itself by relatively large solid particles on which turbulent pulsations have a weaker effect.

## CONCLUSIONS AND PROSPECTS

Studying the origin and evolution of the Solar system and the emergence of various natural conditions on the Earth and other planets is one of the most important trends in modern science. This problem can be solved by performing a series of investigations on the most topical issues of astrophysics, geophysics, and cosmochemistry based on the development of a theory, the generalization and analysis of experimental data, and mathematical modeling. In recent years, the impressive progress in astrophysics, the discoveries of protoplanetary disks and extrasolar planetary systems, and the rapid development of computational mathematics have enhanced the possibilities for comprehensive studies of the physical structure and evolution of the protoplanetary gas–dust disks around young solar-type stars from which the planets are currently believed to be formed.

Adequate cosmogonic models can be constructed by studying the dynamical and thermal evolution of the heterogeneous gas–dust material of a differentially rotating protoplanetary disk with the inclusion of magnetohydrodynamic, turbulent, and radiative effects as well as with the involvement of phase transitions, chemical reactions, and coagulation processes. The aggregate state of the main components of the protoplanetary material, the location of their condensation–sublimation fronts, and, hence, the chemical composition of the planets, their satellites, asteroids, and comets depend on the space–time distribution of thermohydrodynamic parameters for the disk medium. The cosmochemical data obtained by directly studying the extraterrestrial material serve as an important constraint in determining how realistic the models of this kind are.

Unfortunately, a large number of problems related to this trend in research are still outstanding. These primarily include the questions of the early evolution of the Solar system and the reasons why it is unique compared to the known planetary systems near other stars. Developing numerical models for such a dynamical system in which the evolution of an initial protoplanetary cloud sequentially leads to the formation of an accretion gas–dust disk around the young Sun and a compacted dust–gas subdisk is of greatest interest. Thus, the problem of reconstructing the evolution of the protoplanetary gas–dust cloud that surrounded the proton-Sun brings to the fore the following:

—constructing a numerical model for the formation of a dust layer (subdisk) near the midplane of the proto-Sun and studying its flattening mechanisms in a quiet gas and in the presence of turbulence;

—modeling the growth of gravitational instability in a rotating subdisk (when the density of its material becomes higher than a critical value due to its vertical and radial contraction) and the formation and evolution of protoplanetary dust condensations for the zone of inner planets and for the disk periphery;

—modeling the accumulation of the Earth and planets;

—assessing the consequences for the chemical composition of the Earth, planets, asteroids, and comets.

In this paper, the early formation stage of a planetary system, the stage of a protoplanetary gas–dust cloud, is considered as the first step in studying the complex problem of planetary cosmogony. Clearly, numerical simulations of such a cloud are primarily related to the construction of a basic model for a continuum medium with complicated physical–chemical properties that includes, in particular, the magnetohydrodynamic and heat and mass transfer processes simultaneously taking places in a turbulized accretion disk with allowance made for the inertial effects of solid cosmic material particles, radiation, evaporation, condensation, coagulation, and various chemical transformations. Certain aspects of the development of precisely this continuum medium were embodied in this paper, in which the efficient methods of invariant modeling of turbulent flows in multicomponent reactive gas mixtures developed previously (Kolesnichenko and Marov, 1999; Marov and Kolesnichenko, 2002) were further generalized to heterogeneous media. In our view, this study offers prospects for considerably more complete and more realistic modeling of the various processes of the evolution of a differentially rotating protoplanetary gas–dust turbulized disk. Rational schematizations that lead to manageable and solvable equations are particularly needed here to obtain reliable results and to understand them. Our main results include the following:

(1) The formulation of a complete system of equations of two-phase multicomponent mechanics including the relative motion of the phases, coagulation, phase transitions, and radiation intended to formulate and numerically solve the specific model problems of consistently modeling the structure, dynamics, and thermal regime of a protoplanetary accretion disk.

(2) The Favre probability-theoretic averaging of the stochastic equations of heterogeneous mechanics with the goal of phenomenologically describing the turbulent flow of the disk material and the derivation of the defining relations for various turbulent flows needed to close the equations of mean motion.

(3) The development of a semiempirical method for modeling the coefficient of turbulent viscosity in a two-phase disk medium including the inverse effect of the dispersed phase.

(4) The description of the influence of inertial effects of solid particles on turbulence parameters in the disk, in particular, on the additional generation of turbulent energy by large particles, in terms of the model of the medium under consideration.

(5) The development of a parametric method of moments for solving the Smoluchowski intergo-differential coagulation equation for the particle size distribution function based on the fact that the sought-for distribution function belongs to a certain parametric class of distributions.

(6) Analysis of the "regime of limiting saturation" of the subdisk atmosphere by fine dust particles that contributes to the effective settling of solid particles to the midplane.

The results of our numerical solution of the specific problems that reproduce the individual evolutionary stages of a protoplanetary gas–dust cloud based on the model of a disk continuum medium developed here with complicated physical–chemical properties will be presented in ensuing publications.

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## APPENDIX A

### *Solving the Kinetic Coagulation Equation by the Method of Moments*

Let us consider the method of moments for solving the kinetic coagulation equation (33) for the case where the particle size distribution depends on one space coordinate $z$. For the problem considered here, this corresponds to steady motion in a dust layer as particles settle to the midplane of the disk under gravity.

The method of moments consists in reducing the kinetic coagulation equation (33) to a system of ordinary differential equations for the moments

$$M_p = \int_0^\infty U^p f(U, z)\mathrm{d}U \quad (p = 0, 1, 2, \ldots), \tag{A.1}$$

of the distribution function $f(U, z)$. To derive such a system, let us multiply both sides of the simplified equation (33)

$$w_z \frac{\partial f(U,z)}{\partial z} = \frac{1}{2}\int_0^U f(W,z) f(U-W,z) \times K(W, U-W)\mathrm{d}W - f(U,z)\int_0^\infty f(W,z)K(W,U)\mathrm{d}W,$$

where $w_z$ is a constant velocity of dust settling from height $h_{\text{disk}}$ to the midplane of the disk, by $U^p$ and integrate the result over $U$ from 0 to $\infty$; As a result, we obtain

$$w_z \frac{\partial M_0}{\partial z} = w_z \frac{\partial N_\mathrm{d}}{\partial z} = -\frac{1}{2}\int_0^\infty\int_0^\infty K(U,W) f(U,z) f(W,z)\mathrm{d}W\mathrm{d}U, \quad \text{(A.2)}$$

$$w_z \frac{\partial M_1}{\partial z} = w_z \frac{\partial s}{\partial z} = \frac{1}{2}\int_0^\infty\int_0^\infty (W-U)K(U,W) f(U,z) f(W,z)\mathrm{d}W\mathrm{d}U = 0, \quad \text{(A.3)}$$

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$$w_z \frac{\partial M_p}{\partial z} = \int_0^\infty\int_0^\infty K(U,W)[1/2(U+W)^p - U^p] \times f(U,z) f(W,z)\mathrm{d}W\mathrm{d}U \quad (p = 2, 3, \ldots). \quad \text{(A.4)}$$

To express the right-hand sides of the equations of this system in terms of the moments, we must specify the coagulation kernel and make the assumption regarding the form of the distribution function. Let us consider kernels of the type

$$K(U, W) = \Lambda \sum_{j=0}^{K} \beta_j (U^{\alpha-\alpha_j} W^{\alpha_j} + U^{\alpha_j} W^{\alpha-\alpha_j}), \quad \text{(A.5)}$$

where $\Lambda$ is the factor determined by the external conditions under which coagulation takes place. The kernels for which the kinetic equation admits of exact solutions belong to this class of kernels. In addition, formulas of type (A.5) can approximate the kernels that correspond to many coagulation mechanisms in a gas-dust disk, in particular, those analyzed by Kolesnichenko (2001). Indeed, all of the kernels considered by this author are uniform functions for which we may write $K(U, W) = U^\alpha K(1, x)$, where $\alpha$ is the degree of uniformity, $x \equiv W/U$. It is well known that any such function can be approximated by a polynomial:

$$K(U, W) = \Lambda U^\alpha \sum_{j=0}^{K} \beta_j (x^{\alpha_j} + x^{\alpha-\alpha_j}); \quad \text{(A.6)}$$

to determine the unknown coefficients $\beta_j$, we should choose $s$ (interpolation) points $x_i$ whose number is equal to the number of unknown coefficients $\beta_j$ and assume that

$$K(1, x_i) = \Lambda \sum_{j=0}^{K} \beta_j (x_i^{\alpha_j} + x_i^{\alpha-\alpha_j}) \quad (i = 1, 2, \ldots, s). \quad \text{(A.7)}$$

In order that no moments with an order higher than the degree of uniformity of the kernel appear after the subsequent integration of expansion (A.6), the quantity $\alpha_j$ should be defined by the equality $\alpha_j = \alpha j/(K+1)$. Different interpolation polynomials can be obtained at different values of $K$.

Substituting (A.5) in (A.2)–(A.4) yields the following infinite system of equations for the moments:

$$w_z \frac{\partial M_0}{\partial z} = w_z \frac{\partial N_\mathrm{d}}{\partial z} = -\Lambda \sum_{j=0}^{K} \beta_j M_{\alpha-\alpha_j} M_{\alpha_j}, \quad \text{(A.8)}$$

$$w_z \frac{\partial M_1}{\partial z} = w_z \frac{\partial s}{\partial z} = 0, \quad \text{(A.9)}$$

.......................................................

$$w_z \frac{\partial M_p}{\partial z} = \frac{\Lambda}{2}\sum_{j=0}^{K} \beta_j \times \sum_{k=1}^{p-1} C_p^k (M_{\alpha-\alpha_j+p-k} M_{\alpha_j+k} + M_{\alpha-\alpha_j+k} M_{\alpha_j+p-k}) \quad (p = 2, 3, \ldots). \quad \text{(A.10)}$$

To solve this system, whose right-hand sides generally include fractional moments, we must complement it by the coupling equations between the fractional and integer moments. Here, different approaches are also possible: an approach related to the approximation (by Lagrangian polynomials) of fractional moments via integer moments (se, e.g., Loginov, 1979) and a parametric method.

Below, we restrict our analysis to the parametric method, which is based on the fact that the sought-for distribution function $f(U, z)$ belongs to a certain parametric class of distributions. Let us assume, for simplicity, that the distribution $f(U, z)$ remains in the class of distributions to which the initial distribution belongs after coagulation and that only its statistical parameters, the mean, variance, etc., change (with height) as the particles settle to the midplane of the disk. By analogy with the atmospheric aerosol, we choose a two-parameter lognormal distribution as the initial dust particle size (diameter $d$) distribution.

The probability density of the lognormal law depends on the mean $\langle \ln d \rangle$ and variance of the logarithm of the diameter $\sigma_L^2 \equiv \langle (\ln d - \langle \ln d \rangle)^2 \rangle$:

$$p(d; \mu^*, \sigma_L) = \frac{N_d}{\sigma_L d\sqrt{2\pi}} \exp\left\{-\frac{(\ln d - \langle \ln d \rangle)^2}{2\sigma_L^2}\right\} = \frac{N_d}{\sigma_L d\sqrt{2\pi}} \exp\left\{-\frac{\ln^2(d/\mu^*)}{2\sigma_L^2}\right\}. \quad (A.11)$$

The median of the distribution is known to be defined by the relation $\mu^* = \exp(\langle \ln d \rangle)$, while the mean value of the diameter itself and its variance are, respectively,

$$\langle d \rangle = \exp\left(\frac{1}{2}\sigma_L^2 + \ln \mu^*\right), \quad (A.12)$$

$$\sigma^2 \equiv \langle (d - \langle d \rangle)^2 \rangle = \langle d \rangle^2 [\exp \sigma_L^2 - 1]. \quad (A.13)$$

Using the above relations, we can derived formulas for the statistical parameters ($\sigma_L^2$ and $\mu^*$) of the lognormal distribution (A.11) only via the mean particle diameter $\langle d \rangle$ and its relative variance $\beta^2 \equiv \langle (d - \langle d \rangle)^2 \rangle / \langle d \rangle^2$:

$$\sigma_L^2 = \ln(1 + \beta^2), \quad \mu^* = \langle d \rangle / \sqrt{1 + \beta^2}. \quad (A.14)$$

To determine the density of the initial distribution of the dust particle volume $U = (\pi/6)d^3$, we use the formula

$$f(U) = p[d(U)]|\mathrm{d}d/\mathrm{d}U| \quad (A.15)$$

(which is valid for a strictly increasing function $U = U(d)$ of the random variable $d$ (Khan and Shapiro, 1969)) and distribution (A.11); as a result, we obtain

$$f(U; \sigma_L, \mu) = \frac{N_d}{3\sqrt{2\pi}\sigma_L U} \exp\left[-\frac{\ln^2(U/\mu)}{18\sigma_L^2}\right], \quad (A.16)$$

where $\mu = (\pi/6)\mu^{*3}$.

Let the coagulation in the disk do not change this distribution and only the parameters $\mu(z)$ and $\sigma_L^2(z)$ change with height. We introduce the moments of the lognormal distribution

$$M_p(z) = \frac{N_d}{3\sqrt{2\pi}\sigma_L(z)} \times \int_0^\infty U^{p-1} \exp\left[-\frac{\ln^2[U/\mu(z)]}{18\sigma_L^2(z)}\right] \mathrm{d}U. \quad (A.17)$$

According to Lee (1983), the following representation is valid for any moment of order $p$:

$$M_p = M_1 \mu^{p-1} \exp[3/2(p^2 - 1)\sigma_L^2], \quad (A.18)$$

$$M_1 = s = \mathrm{const}, \quad (A.19)$$

which allows the fractional moments in (A.8)–(A.10) to be expressed in terms of $M_1$, $\mu$, and $\sigma_L^2$. As a result, we obtain the following parametric system of two ordinary differential equations (the number of equations must be equal to the number of unknown coefficients) to determine the parameters $\mu(z)$ and $\sigma_L^2(z)$ from given boundary values of $\mu(h_{\mathrm{disk}})$ and $\sigma_L^2(h_{\mathrm{disk}})$:

$$\begin{aligned} w_z \frac{\partial M_0}{\partial z} &= -\Lambda \sum_{j=0}^{K} \beta_j M_{\alpha - \alpha_j} M_{\alpha_j} \\ &= -\Lambda M_1^2 \mu^{\alpha - 2} \sum_{j=0}^{K} \beta_j \exp[3/2[\alpha_j^2 + (\alpha - \alpha_j)^2 - 2]\sigma_L^2] = \\ &= -\Lambda \mu^\alpha M_0^2 \sum_{j=0}^{K} \beta_j \exp[3/2[\alpha_j^2 + (\alpha - \alpha_j)^2]\sigma_L^2], \end{aligned} \quad (A.20)$$

$$\begin{aligned} w_z \frac{\partial M_2(z)}{\partial z} &= 2\Lambda \sum_{j=0}^{K} \beta_j M_{\alpha - \alpha_j + 1} M_{\alpha_j + 1} \\ &= 2\Lambda M_1^2 \mu^\alpha \sum_{j=0}^{K} \beta_j \exp[3/2[(\alpha_j + 1)^2 + (\alpha - \alpha_j + 1)^2 - 2]\sigma_L^2] \\ &= 2\Lambda M_1^2 \mu^\alpha \sum_{j=0}^{K} \beta_j \exp[3/2[(\alpha_j + 1)^2 + (\alpha - \alpha_j + 1)^2 - 2]\sigma_L^2] \\ &= 2\Lambda \mu^{\alpha+2} \sum_{j=0}^{K} \beta_j \exp[3/2[\alpha_j^2 + (\alpha - \alpha_j)^2]\sigma_L^2]. \end{aligned} \quad (A.21)$$

$$\begin{aligned} N_d \equiv M_0 &= s\mu^{-1} \exp(-3/2\sigma_L^2); \\ M_2 &= s\mu \exp(9/2\sigma_L^2). \end{aligned} \quad (A.22)$$

This parametric system of nonlinear equations can be solved only numerically. Our numerical simulations for the problem of dust settling to the midplane of the disk will be presented in a special paper. Here, we note that the change in the mean number of particles with height $N_d(z)$ can be estimated by assuming that the variance

$\sigma_L^2$ remains constant. In this case, restricting our selves to the first two moments, we find from (A.21) that

$$w_z\frac{\partial N_d}{\partial z} = -\Lambda\mu^\alpha N_d^2\sum_{j=0}^K\beta_j\exp[3/2[\alpha_j^2+(\alpha-\alpha_j)^2]\sigma_L^2]. \quad (A.23)$$

The solution of this equation obtained using the boundary condition $N_d(h_{disk}) = N_{d,h_{disk}} = s/\tilde{U}(h_{disk})$ is

$$N_d(z) = \frac{s}{\tilde{U}(h_{disk})}\frac{1}{1+q[(h_{disk}-z)/w_z]}, \quad (A.24)$$

where

$$q = \Lambda\mu^\alpha\frac{s}{\tilde{U}(h_{disk})} \times\sum_{j=0}^K\beta_j\exp[3/2[\alpha_j^2+(\alpha-\alpha_j)^2]\sigma_L^2]. \quad (A.25)$$

Here, $\tilde{U}(h_{disk}) = (\pi/6)\langle d\rangle^3 = \mu(h_{disk})\exp(3/2\sigma_L^2)$ is the upper limit of the mean volume (this formula follows from (A.14)). Thus, using the relation $\tilde{U}(z) = s/N_d$, we can determine the change in the mean particle volume with height. For relatively low $z$ (i.e., when the particles are already near the midplane of the disk), $q[(h_{disk} - z)/w_z] \gg 1$, it follows from (A.25) that

$$N_d(z) = \left\{\Lambda\mu^\alpha[(h_{disk}-z)/w_z] \times\sum_{j=0}^K\beta_j\exp[3/2[\alpha_j^2+(\alpha-\alpha_j)^2]\sigma_L^2]\right\}^{-1}. \quad (A.26)$$

We see from this expression that the mean number of particles in the system for a fairly long coagulation time ceases to depend on their initial distribution, i.e., as it were, “forgets its past, and can be described by a universal function whose form is determined only by the coagulation kernel. A similar analysis can also be performed with other possible dust particle volume distributions in a coagulating turbulent flow, for example, with the Gamma distribution.

## APPENDIX B

### *Cylindrical Coordinates*

Here, for convenience, we present, in cylindrical coordinates $r$, $\varphi$, and $z$ (for the axisymmetric case, $\partial/\partial\varphi = 0$), the expressions for the various operators in the above equations of heterogeneous mechanics that act on

(1) scalars:

$$\frac{d\mathcal{B}}{dt} \equiv \frac{\partial\mathcal{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathcal{B} = \frac{\partial\mathcal{B}}{\partial t} + u_r\frac{\partial\mathcal{B}}{\partial r} + u_z\frac{\partial\mathcal{B}}{\partial z}, \quad \nabla\mathcal{B} = \mathbf{i}_r\frac{\partial\mathcal{B}}{\partial r} + \mathbf{i}_z\frac{\partial\mathcal{B}}{\partial z}, \quad (B.1)$$

$$\nabla^2\mathcal{B} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\mathcal{B}}{\partial r}\right) + \frac{\partial^2\mathcal{B}}{\partial z^2}; \quad (B.2)$$

(2) vectors:

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial(r\mathcal{A}_r)}{\partial r} + \frac{\partial\mathcal{A}_z}{\partial z}, \quad (B.3)$$

$$\nabla\mathbf{A} = \mathbf{i}_r\mathbf{i}_r\frac{\partial\mathbf{A}_r}{\partial r} + \mathbf{i}_r\mathbf{i}_\varphi\frac{\partial\mathbf{A}_\varphi}{\partial r} + \mathbf{i}_r\mathbf{i}_z\frac{\partial\mathbf{A}_z}{\partial r} + \mathbf{i}_\varphi\mathbf{i}_r\frac{\mathbf{A}_\varphi}{r} + \mathbf{i}_\varphi\mathbf{i}_\varphi\frac{\mathbf{A}_r}{r} + \mathbf{i}_z\mathbf{i}_r\frac{\partial\mathbf{A}_r}{\partial z} + \mathbf{i}_z\mathbf{i}_\varphi\frac{\partial\mathbf{A}_\varphi}{\partial z} + \mathbf{i}_z\mathbf{i}_z\frac{\partial\mathbf{A}_z}{\partial z}; \quad (B.4)$$

(3) dyads:

$$\mathbf{P} = \mathbf{i}_r\mathbf{i}_rP_{rr} + \mathbf{i}_r\mathbf{i}_\varphi P_{r\varphi} + \mathbf{i}_r\mathbf{i}_zP_{rz} + \mathbf{i}_\varphi\mathbf{i}_rP_{\varphi r} + \mathbf{i}_\varphi\mathbf{i}_\varphi P_{\varphi\varphi} + \mathbf{i}_\varphi\mathbf{i}_zP_{\varphi z} + \mathbf{i}_z\mathbf{i}_rP_{zr} + \mathbf{i}_z\mathbf{i}_\varphi P_{z\varphi} + \mathbf{i}_z\mathbf{i}_zP_{zz}, \quad (B.5)$$

$$\nabla\cdot\mathbf{P} = \mathbf{i}_r\left[\frac{1}{r}\frac{\partial(rP_{rr})}{\partial r} - \frac{\partial P_{zr}}{\partial z} - \frac{P_{\varphi\varphi}}{r}\right] + \mathbf{i}_\varphi\left[\frac{1}{r}\frac{\partial(rP_{r\varphi})}{\partial r} + \frac{\partial P_{z\varphi}}{\partial z} + \frac{P_{\varphi r}}{r}\right] + \mathbf{i}_z\left[\frac{1}{r}\frac{\partial(rP_{rz})}{\partial r} + \frac{\partial P_{zz}}{\partial z}\right]. \quad (B.6)$$

For the deformation tensors and the deformation rate tensor, we then have

$$\mathbf{D} \equiv \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\text{transp}})$$
$$= \mathbf{i}_r\mathbf{i}_r\frac{\partial u_r}{\partial r} + \mathbf{i}_r\mathbf{i}_\varphi\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \mathbf{i}_r\mathbf{i}_z\frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \mathbf{i}_\varphi\mathbf{i}_r\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \mathbf{i}_\varphi\mathbf{i}_\varphi\frac{u_r}{r} + \mathbf{i}_\varphi\mathbf{i}_z\frac{1}{2}\frac{\partial u_\varphi}{\partial z} + \mathbf{i}_z\mathbf{i}_r\frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \mathbf{i}_z\mathbf{i}_\varphi\frac{1}{2}\frac{\partial u_\varphi}{\partial z} + \mathbf{i}_z\mathbf{i}_z\frac{\partial u_z}{\partial z}, \quad (B.7)$$

$$\overset{\circ}{\mathbf{D}} \equiv \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\text{transp}}) - \frac{1}{3}\mathbf{I}\nabla\cdot\mathbf{u}$$
$$= \mathbf{i}_r\mathbf{i}_r\left(\frac{\partial u_r}{\partial r} - \frac{1}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right) + \mathbf{i}_r\mathbf{i}_\varphi\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \mathbf{i}_r\mathbf{i}_z\frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \mathbf{i}_\varphi\mathbf{i}_r\frac{1}{2}\left(\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}\right) + \mathbf{i}_\varphi\mathbf{i}_\varphi\left(\frac{u_r}{r} - \frac{1}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right) + \mathbf{i}_\varphi\mathbf{i}_z\frac{1}{2}\frac{\partial u_\varphi}{\partial z} + \mathbf{i}_z\mathbf{i}_r\frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) + \mathbf{i}_z\mathbf{i}_\varphi\frac{1}{2}\frac{\partial u_\varphi}{\partial z} + \mathbf{i}_z\mathbf{i}_z\left(\frac{\partial u_z}{\partial z} - \frac{1}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right). \quad (B.8)$$

The two-point Gibbs multiplication is an operator that is widely used in hydrodynamics. According to the Gibbs notation, if **a**, **b**, **c**, and **d** are arbitrary vectors, then **ab** : **cd** = (**a** · **c**)(**b** · **d**). In particular, for unit vectors, we can write

$$\mathbf{i}_j\mathbf{i}_k : \mathbf{i}_l\mathbf{i}_m = (\mathbf{i}_j \cdot \mathbf{i}_l)(\mathbf{i}_k \cdot \mathbf{i}_m) = \delta_{jl}\delta_{km}; \tag{B.9}$$

for two dyads, we then have

$$\mathbf{D}^{(1)} : \mathbf{D}^{(2)} = \left(\sum_j\sum_k \mathbf{i}_j\mathbf{i}_k D_{jk}^{(1)}\right) : \left(\sum_l\sum_m \mathbf{i}_l\mathbf{i}_m D_{lm}^{(2)}\right) \tag{B.10}$$

$$= \sum_j\sum_k\sum_l\sum_m \delta_{jl}\delta_{km} D_{jk}^{(1)} D_{lm}^{(2)} = \sum_j\sum_k D_{jk}^{(1)} D_{jk}^{(2)}$$

or

$$2\mathring{\mathbf{D}} : \mathring{\mathbf{D}} = 2D_{rr}^2 + 2D_{\varphi\varphi}^2 + D_{zz}^2 + 4D_{r\varphi}^2 + 4D_{rz}^2 + 4D_{z\varphi}^2 - 2/3(\nabla \cdot \mathbf{u})^2. \tag{B.11}$$

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